Optimality of Trunk Reservation for an M/M/k/N Queue with Several Customer Types and Holding Costs

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Abstract

We study optimal admission to an M/M/k/N queue with several customer types. The reward structure consists of revenues collected from admitted customers and holding costs, both of which depend on customer types. This paper studies average rewards per unit time and describes the structures of stationary optimal, canonical, bias optimal, and Blackwell optimal policies. Similar to the case without holding costs, bias optimal and Blackwell optimal policies are unique, coincide, and have a trunk reservation form with the largest optimal control level for each customer type. Problems with one holding cost rate have been studied previously in the literature.

1 Introduction

We consider an M/M/k/N queue with m customer types, where $m \geq 1$. Customers of type j , $j = 1, 2, \dots, m$, arrive at the system according to independent Poisson processes with rates λ_j , where $0 < \lambda_j < \infty$. When a customer arrives at the system, its type

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becomes known. There are k identical servers in the system, where $k \geq 1$. The service times are independent, do not depend on the customer types, and are exponentially distributed with rate μ , where $0 < \mu < \infty$. When there are n customers in the system, the total service rate of the system is μ_n , where $\mu_n = n\mu$ for $n = 0, 1, \dots, k-1$, and $\mu_n = k\mu$ for $n = k, k+1, \dots, N$. Moreover, there is no preemption for customers. The queue follows the first-in-first-out (FIFO) rule.

At the arrival epochs, the system manager decides whether an arrival can enter the system or not. If a customer sees less than k customers in the system and is admitted, the customer goes to a free server immediately. If a customer sees at least k customers in the system and is admitted, the customer waits in the queue for service. If there are N customers in the system, the system is full and all the arrivals are rejected. Upon admitting a customer, the system collects a positive reward which depends on the customer type and incurs a nonnegative random holding cost which depends on the customer type and the number of customers in the system the admitted customer sees. Let $r_j(n)$ be the net reward collected by the system if a customer of type j sees n customers in the system and is admitted. The net rewards $r_j(n)$ are positive constants for $n = 0, 1, \dots, k-1$, and are nonincreasing in $n = k-1, k, \dots, N-1$. The dependence of $r_j(n)$ on n reflects the fact that net rewards depend on waiting times. Furthermore, we assume that different customer types have different reward functions. When an arrival is rejected, the system does not collect any reward. The objective is to maximize the long-run average reward. We call a policy, that maximizes the long-run average reward, an optimal policy. In addition to average reward optimality, we also consider three more selective policies: canonical, bias optimal, and Blackwell optimal policies. Below is the definition of a trunk reservation policy.

Definition 1.1 *A policy ϕ is called a trunk reservation policy (TRP) if there are m control levels M_j^ϕ , $j = 1, \dots, m$, such that a type j arrival is admitted to the system if and only if the customer sees less than M_j^ϕ customers in the system and a type j*

arrival is rejected if the customer sees at least M_j^ϕ customers in the system.

Optimal admission control problem has been extensively studied in the literatures, but the results are limited to the models without holding costs or with holding costs that do not depend on customer types. Miller [11] considered m types of customers for an M/M/k/loss queue without holding costs and ordered rewards $r_1 > r_2 > \dots > r_m > 0$. He proved the existence of an optimal TRP f with $k = M_1^f \geq M_2^f \geq \dots \geq M_{m-1}^f \geq M_m^f$. Feinberg and Reiman [4] considered general birth and death processes and studied a problem with a constraint. In particular, they extended Miller's [11] result to an M/M/k/N queue without holding costs. In addition, Feinberg and Reiman [4] showed that any randomized stationary optimal policy for a problem without holding costs is a randomized TRP. Lewis et al. [10] provided a simpler proof of this fact for (nonrandomized) stationary policies. Problems with more general constraints than in [4] were studied by Fan-Orzechowski and Feinberg [2, 3].

Several optimal admission control problems with holding costs have been studied. Naor [13] proved the existence of an optimal TRP for an M/M/1 queue with one customer type and linear holding costs. Knudsen [9] extended Naor's result to an M/M/k queue. Yechiali [18, 19] extended Naor's and Knudsen's results to GI/M/1 and GI/M/k queues with one customer type and linear holding costs. Stidham [15] proved the existence of an optimal TRP for a GI/M/1 queue with one customer type and convex nondecreasing holding costs; see also [16]. Johansen [8] extended Stidham's result to a GI/M/k queue. Helm and Waldman [7] extended Stidham's [15] and Johansen's [8] results to GI/M/k queues with batch arrivals, convex nondecreasing holding costs, and changing environment.

For a Markov Decision Process (MDP) with finite state and action sets, the optimal policies found by Howard's policy iteration algorithm satisfy optimality equations and such policies are called canonical. Blackwell [1] proved the existence of a stationary policy that optimizes the expected total discounted reward for all discount factors

$\beta \in [\beta^*, 1)$ for some $\beta^* \in [0, 1)$. Such policies are called Blackwell optimal. A policy is called bias optimal if the difference of the expected total discounted rewards between this policy and a Blackwell optimal policy tends to 0 as $\beta \rightarrow 1$. Veinott [17] modified Howard's [5] policy iteration algorithm to find a bias optimal policy. For continuous time problems, the values of the discount factor β close to 1 correspond to the values of the discount rate α close to 0, because essentially $\beta = e^{-\alpha}$.

For an M/M/1 queue with one customer type and convex increasing holding costs, Haviv and Puterman [6] showed that there is either one optimal TRP or two optimal TRPs. If there are two optimal TRPs, the difference between their control levels is 1. Haviv and Puterman [6] also showed that the optimal TRP with the larger optimal control level is the unique bias optimal policy and, therefore, it is also Blackwell optimal. Lewis et al. [10] proved that a similar result holds for an M/M/k/N queue with several customer types and without holding costs.

In this paper, we prove the existence of an optimal TRP for an M/M/k/N queue with holding costs depending on customer types. Unlike the case of zero holding costs, where any stationary optimal policy is a TRP, a stationary optimal policy may not be a TRP, but it becomes a TRP after decisions are changed at appropriate states. In addition, we provide complete descriptions of the classes of stationary optimal policies, optimal TRPs, canonical policies, bias optimal policies, and Blackwell optimal policies. We show that the bias optimal policy for our problem is unique and it is also Blackwell optimal. Similar to the cases of one customer type [6] and several customer types without holding costs [10], the bias optimal policy selects the largest optimal control level for each customer type. For all customer types, except at most one, there exist at most two optimal control levels. Under a certain condition, there is one customer type, for whom there are more than two optimal control levels. For this customer type, the largest optimal control level is N .

2 Problem Formulation

Consider the M/M/k/N queue defined in the introduction. Following the definitions, we formulate $r_j(n)$ and the continuous-time Markov decision process (CTMDP) for our problem. Let R_j be the reward an admitted type j customer pays for the service and $h_j(t) \geq 0$ be the rate the system incurs holding costs to type j customers, where t is the time passed after the customer's admission.

Let D_n , $n = 0, 1, \dots, N - 1$, be the delay for an admitted customer seeing n customers in the system at the arrival time, and W_n be the waiting time for such a customer. In other words, if a customer is admitted when there are n customers in the system, the admitted customer spends W_n units of time in the system and D_n units of time in the queue waiting until the service begins.

If type j arrival is admitted when there are n customers in the system, the expected holding cost $H_j^d(n)$ incurred during the time the customer waits for service in the queue is

$$H_j^d(n) = E \int_0^{D_n} h_j(t) dt,$$

and the expected holding cost $H_j^s(n)$ incurred during the time the customer spends in the system is

$$H_j^s(n) = E \int_0^{W_n} h_j(t) dt.$$

The reward functions $r_j(n) = R_j - H_j(n)$, $n = 0, 1, \dots, N - 1$, where either $H_j(n) = H_j^d(n)$ or $H_j(n) = H_j^s(n)$ depending on a particular problem. In the either case, the functions $r_j(n)$ possess some natural properties. For example, $r_j(n) = r_j(0)$ for $n = 1, \dots, k - 1$. If $r_j(n) = R_j - H_j^d(n)$, then $r_j(0) = R_j$. If $r_j(n) = R_j - H_j^s(n)$, then $r_j(0) = R_j - H_j^s(0)$. In the either case, it is natural to assume that $r_j(0) > 0$. Otherwise, type j customers should be always rejected. If $h_j(t) = 0$ for all $t \geq 0$, then $r_j(n) = r_j(0)$ for all n . If h_j are measurable functions such that $a \leq h_j(t) \leq b$ for two finite positive constants a and b , then $\frac{a}{k\mu} \leq r_j(n - 1) - r_j(n) \leq \frac{b}{k\mu}$ for $n \geq k$. We

assume the following broad and natural condition throughout this paper.

Condition 2.1 *For each $j = 1, 2, \dots, m$, the reward function $r_j(n)$ satisfies the following conditions: (a) $r_j(n) = r_j(0) > 0$ for all $n = 0, 1, \dots, k-1$, (b) either $r_j(n) = r_j(0)$ for all $n = k, k+1, \dots, N-1$ or $r_j(n) < r_j(n-1)$ for all $n = k, k+1, \dots, N-1$, and (c) different customer types have different reward functions, that is, if $j \neq l$, then $r_j(n) \neq r_l(n)$ for some $n = 0, 1, \dots$, where $j, l = 1, \dots, m$.*

We formulate our problem as a CTMDP with the state space $S = \{0, 1, \dots, N\}$. State n means there are n customers in the system. $A(n)$ is the set of actions available at state n and $A = \bigcup_{n=0}^N A(n)$. An action $a \in A(n)$ is the set of the customer types to be admitted at state n and $a = \emptyset$ means no customer is accepted. Thus, $A(n)$, $n = 0, 1, \dots, N-1$, contains 2^m actions, which are all the possible subsets of $\{1, 2, \dots, m\}$, and $A(N) = \{\emptyset\}$. Each action $a \in A(n)$ defines a transition intensity $\lambda_n^a = \sum_{j \in a} \lambda_j$ and an expected reward per unit time $R(n, a) = \sum_{j \in a} \lambda_j r_j(n)$. Since S and A are finite, there exists a (nonrandomized) stationary optimal policy for our problem. Thus, in the rest of this paper, we focus on the set of all stationary policies and let F be this set. For any $f \in F$, we denote by f_n the action that f chooses from $A(n)$ at state n . We also define $f(n, j)$ as follows:

$$f(n, j) = \begin{cases} \{accept\}, & \text{if } j \in f_n; \\ \{reject\}, & \text{otherwise.} \end{cases}$$

Moreover, the policy f defines a vector of expected reward per unit time $R(f)$ whose $(n+1)$ -st component is $R(n, f_n)$ and an infinitesimal matrix $Q(f)$ whose (n, l) element is $q(l|n, f_n)$, where $q(n+1|n, f_n) = \lambda_n^f$ for $n = 0, 1, \dots, N-1$, $q(n-1|n, f_n) = \mu_n$ for $n = 1, 2, \dots, N$, $q(n|n, f_n) = -\mu_n - \lambda_n^f$ for $n = 0, 1, \dots, N$, and $q(l|n, f_n) = 0$, otherwise. Furthermore, let $P_{nl}(t, f)$ be the probability that the process is at state l at time t , given that the process started at state n at time 0 under policy f . We let

$P(t, f)$ be the corresponding matrix. Then $P(t, f)$ converges to the steady probability matrix $P^*(f)$ as $t \rightarrow \infty$. We denote v_n^f the average reward per unit time, given that the process starts at state n under f , and V^f the vector whose $(n+1)$ -st element is v_n^f , $n = 0, 1, \dots, N$. Then

$$v_n^f = \liminf_{T \rightarrow \infty} T^{-1} E_n^f \int_0^T R(x_t, a_t) dt,$$

where $R(x_t, a_t)$ is the expected reward per unit time collected when the process is at state x_t and action a_t is chosen at time t , and

$$V^f = \lim_{T \rightarrow \infty} T^{-1} \int_0^T P(t, f) R(f) dt = P^*(f) R(f).$$

Our goal is to find an optimal policy that maximizes the vector V^f . Note that there is only one recurrent class under any policy from F . Such CTMDPs are called unichain. Since the CTMDP is unichain, all rows of $P^*(f)$ are equal to the unique steady state probability vector $(\pi_0^*(f), \pi_1^*(f), \dots, \pi_N^*(f))$. Thus, all elements of V^f are equal and $v_n^f = v^f = \sum_{s=0}^{N-1} \pi_s^*(f) R(s, f_s)$, $n = 0, 1, \dots, N$. A policy $f \in F$ is optimal if $v^f \geq v^\phi$ for all $\phi \in F$. Let F^* be the set of stationary optimal policies and v^* be the average reward per unit time under any policy from F^* . For a unichain CTMDP, a policy f is called canonical if and only if there exists a function y_j^f such that

$$v^f = R(i, f_i) + \sum_{j=0}^N q(j|i, f_i) y_j^f = \max_{a \in A(i)} \{R(i, a) + \sum_{j=0}^N q(j|i, a) y_j^f\} \quad (1)$$

for $i = 0, 1, \dots, N$. For a unichain CTMDP with finite states and action sets, a canonical policy always exists, is optimal, and Howard's policy iteration algorithm computes it; see [20]. Similar to discrete-time MDPs, an optimal policy may not be canonical.

3 Average Reward Optimality

The results of Feinberg and Reiman [4, Theorem 3.1] imply that, for any M/M/k/N queue with m customer types and without holding costs, any stationary optimal policy is a TRP; see also [10] for a simpler proof of this fact. In addition, Feinberg and Reiman [4] showed that, under any optimal policy f , if $R_l > R_k$, then $M_l^f \geq M_k^f$, and, if $R_i = \max_{j=1,\dots,m} R_j$, then $M_i^f = N$. In this section, we prove the existence of an optimal TRP for the problem with holding costs and obtain the structure of the canonical policies. Contrary to the results for problems without holding costs [4, 10], a stationary optimal policy may not be a TRP; see Examples 3.1 and 3.2. However, under some conditions, we show that an optimal policy has a closely related form which we call essential trunk reservation. Moreover, because $r_j(n)$ depend on the customer types and the states, we show in Example 3.1 that $r_i(0) > r_j(0)$ does not imply $M_i^f \geq M_j^f$ under an optimal TRP f .

In order to compute an optimal policy for our problem, consider the following equations for a CTMDP [11]:

$$V^f = P^*(f) \times R(f), \quad (2)$$

$$R(f) + Q(f) \times y^f = V^f, \quad (3)$$

$$P^*(f) \times y^f = 0, \quad (4)$$

where y^f is the bias vector and $f \in F$. According to [11],

$$y^f = \int_{t=0}^{\infty} (P(t, f) - P^*(f)) R(f) dt.$$

We denote by y_n^f the $(n+1)$ -st element of y^f , $n = 0, 1, \dots, N$. Let $\nabla y_n^f = y_n^f - y_{n+1}^f$ for $n = 0, 1, \dots, N-1$.

Miller [11] presented a version of Howard's policy iteration algorithm [5] for the

optimal admission problem for any M/M/k/loss queue with several customer types. In that version, Miller [11] considered relative bias functions ∇y_n^f instead of bias functions y_n^f . The same can be done for birth-and-death processes and reward functions depending on the states. In addition, for any $f \in F$ and $n = 0, 1, \dots, N$, we define

$$H^f(n, z) = \sum_{j \in f_n} \lambda_j (r_j(n) - z),$$

where $H^f(n, z) = 0$ if $f_n = \emptyset$. In particular, $H^f(N, z) = 0$. Also, we let $\nabla y_{-1}^f = 0$ for any $f \in F$. Below we present a modification of Miller's algorithm [11] for an M/M/k/N queue with reward functions depending on states and customer types.

Algorithm 1 *Policy Iteration Algorithm*

1. Choose a policy $f \in F$.
2. For policy f , compute v^f and ∇y_n^f , $n = 0, 1, \dots, N-1$, by solving linear equations

$$H^f(n, \nabla y_n^f) + \mu_n \times \nabla y_{n-1}^f = v^f, \quad n = 0, 1, 2, \dots, N. \quad (5)$$

3. a. If $H^f(n, \nabla y_n^f) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^f)\}$ for $n = 0, 1, \dots, N-1$, then f is an optimal policy. Stop.
- b. For all $n = 0, 1, \dots, N-1$ such that $H^f(n, \nabla y_n^f) < \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^f)\}$, change f_n to a^n such that $\sum_{j \in a^n} \lambda_j (r_j(n) - \nabla y_n^f) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^f)\}$. Go to Step 2. ■

In order to compute the relative bias functions ∇y_n^f , Miller [11] transformed (3) into (5). The following statements are the descriptions for a canonical policy.

Proposition 3.1 *A policy f is canonical if and only if one of the following equivalent statements hold:*

- (a) $H^f(n, \nabla y_n^f) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^f)\}$ for all $n = 0, 1, \dots, N-1$;

(b) $H^f(n, \nabla y_n^f) = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_n^f)^+$ for all $n = 0, 1, \dots, N-1$;

(c) $f(n, j) = \{accept\}$ if $r_j(n) > \nabla y_n^f$ and $f(n, j) = \{reject\}$ if $r_j(n) < \nabla y_n^f$.

Proof: (a)–(b) According to [11], (5) is the same as (3). Thus, from (1), a policy f is canonical if and only if

$$\begin{aligned} v^* &= H^f(n, \nabla y_n^f) + \mu_n \nabla y_{n-1}^f = \max_{a \in A(n)} \left\{ \sum_{j \in a} \lambda_j(r_j(n) - \nabla y_n^f) + \mu_n \nabla y_{n-1}^f \right\} \\ &= \max_{a \in A(n)} \left\{ \sum_{j \in a} \lambda_j(r_j(n) - \nabla y_n^f) \right\} + \mu_n \nabla y_{n-1}^f = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_n^f)^+ + \mu_n \nabla y_{n-1}^f. \end{aligned}$$

Therefore, f is canonical if and only if $H^f(n, \nabla y_n^f) = \max_{a \in A(n)} \{ \sum_{j \in a} \lambda_j(r_j(n) - \nabla y_n^f) \} = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_n^f)^+$ for $n = 0, 1, \dots, N-1$.

(c) According to (b), if $r_j(n) - \nabla y_n^f > 0$, then $f(n, j) = \{accept\}$. On the other hand, if $r_j(n) < \nabla y_n^f$, then $(r_j(n) - \nabla y_n^f)^+ = 0$ and, thus, $f(n, j) = \{reject\}$. ■

Let $n^f = \min\{n = 1, \dots, N : f_n = \emptyset\}$, where $f \in F$. Then, under policy f , the recurrent class $R^f = \{0, 1, \dots, n^f\}$ and the set of transient states $T^f = \{n^f + 1, n^f + 2, \dots, N\}$. This is true because of the following reasons: (i) state 0 is accessible from any state $n = 0, 1, \dots, N$, and, therefore, it is recurrent; (ii) any two states in R^f communicate; (iii) any state in T^f is not accessible from state n^f . Note that $T^f = \emptyset$ if and only if $n^f = N$. For problems without holding costs, $n^f = N$ if $f \in F^*$ [4, 10].

Theorem 3.1 *Every $f \in F^*$ always accepts type l customers with $r_l(0) = \max_{j=1,2,\dots,m} r_j(0)$ when there are less than k customers in the system. Thus, $n^f \geq k$ for $f \in F^*$.*

Proof: Assume $f \in F^*$ reject all customer types at state n , $n < k$. Then f is also optimal for an M/M/k/loss queue. However, according to [4, 11, 10], f always accepts type l customers with $r_l(0) = \max_{j=1,2,\dots,m} r_j(0)$ if there is at least one empty server. This contradiction implies this theorem. ■

For the problem without holding costs, all optimal policies are TRPs [4, 10] and, thus, all canonical policies are TRPs. In the case with holding costs, $r_j(n)$ may decrease

with respect to n for $n \geq k - 1$, and it is possible that all customer types shall be rejected, even if there is available space in the system. Thus, the corresponding Markov chain may contain transient states. If an optimal TRP is changed at these states, then it remains optimal, but it is not a TRP anymore. The following example illustrates this possibility.

Example 3.1 Consider an M/M/1/4 queue with two customer types, where $\lambda_1 = \lambda_2 = \mu = 1$, $r_1(0) = 10$, $r_1(1) = 6$, $r_1(2) = 2$, $r_1(3) = -2$, $r_2(0) = 9$, $r_2(1) = 8$, $r_2(2) = 6$, and $r_2(3) = 3$. Consider a policy $d \in F$ with $d_0 = \{1, 2\}$, $d_1 = \{2\}$, $d_2 = \emptyset$, $d_3 = \{2\}$, and $d_4 = \emptyset$. By solving (5), $v^d = 7$. Now consider another policy $f \in F$, with $f_0 = \{1, 2\}$, $f_1 = \{2\}$, and $f_2 = f_3 = f_4 = \emptyset$. By solving (5), $v^f = 7$. Both d and f are optimal. However, f is a canonical policy and a TRP with $M_1^f = 1 < 2 = M_2^f$, but d is neither a canonical policy nor a TRP. ■

The policies from Example 3.1 satisfy the properties described in the following lemma.

Lemma 3.1 *Let $f \in F^*$. Then the following statements hold for $j = 1, 2, \dots, m$ and $n < N$.*

- (i) *if $r_j(n) > \nabla y_n^f$ and $n \in R^f$, then $f(n, j) = \{\text{accept}\}$;*
- (ii) *if $r_j(n) < \nabla y_n^f$ and $n \in R^f$, then $f(n, j) = \{\text{reject}\}$;*
- (iii) *if $g(n, j) = f(n, j)$ for all $n = 0, 1, \dots, N - 1$ such that $r_j(n) \neq \nabla y_n^f$, then $\nabla y_n^g = \nabla y_n^f$ for all $n = 0, 1, \dots, N - 1$ and $g \in F^*$;*
- (iv) *$H^f(n, \nabla y_n^f) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^f)\} \geq 0$ for all $n \in R^f$.*

Proof: See Appendix.

Below are the definition of the restriction of a policy and the definition of an essential trunk reservation policy.

Definition 3.1 We call $f^R \in F$ the restriction of policy $f \in F$ if

$$f_n^R = \begin{cases} f_n, & \text{if } n \in R^f; \\ \emptyset, & \text{if } n \in T^f. \end{cases}$$

A policy $f \in F$ is called an essential trunk reservation policy (ETRP) if $f^R \in F$ is a TRP.

Lemma 3.2 $v^f = v^{f^R}$ for any $f \in F$, and, therefore, $f \in F^*$ if and only if $f^R \in F^*$.

Proof: Since $n^{f^R} = n^f$, we have that $R^{f^R} = R^f = \{0, 1, \dots, n^f\}$ and $T^{f^R} = T^f = \{n^f + 1, \dots, N\}$. The policies f and f^R coincide in $\{0, 1, \dots, n^f\}$. Thus, $v^f = v^{f^R}$. ■

Let C^* be the set of all canonical policies, E^* be the set of all optimal ETRPs, and T^* be the set of all optimal TRPs. Then the following theorem links the sets of optimal policies F^* , canonical policies C^* , optimal ETRPs E^* , and optimal TRPs T^* .

Theorem 3.2 A policy $f \in F^*$ if and only if $f^R \in C^*$. In addition, $T^* \subseteq E^* \subseteq F^*$. In the case with zero holding costs, $T^* = C^* = E^* = F^*$.

Proof: If $f^R \in C^*$, then $f^R \in F^*$. From Lemma 3.2, $f \in F^*$. If $f \in F^*$, then, from Lemma 3.2, $f^R \in F^*$. Now let $n^f = N$. Then Lemma 3.1(iv) implies that $H^{f^R}(n, \nabla y_n^{f^R}) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^{f^R})\}$ for $n = 0, 1, \dots, N-1$. Proposition 3.1(a) implies $f^R \in C^*$. If $n^f \leq N-1$, then Lemma 3.1 implies $\nabla y_{n^f}^{f^R} \geq \max_{j=1,2,\dots,m} r_j(n^f)$. Since $f_n^R = \emptyset$ for $n \geq n^f$, formula (5) implies $\nabla y_{n^f-1}^{f^R} = \nabla y_{n^f}^{f^R} = \dots = \nabla y_{N-1}^{f^R} = \frac{v^*}{k\mu} \geq \max_{j=1,2,\dots,m} r_j(n^f)$. Since $r_j(n)$ are nonincreasing in $n = 0, 1, \dots, N-1$, we have $\nabla y_n^{f^R} \geq \max_{j=1,2,\dots,m} r_j(n)$ for $n \in T^f = \{n^f+1, n^f+2, \dots, N\}$ and $H^{f^R}(n, \nabla y_n^{f^R}) = 0 = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^{f^R})\}$ for $n \in T^f$. Lemma 3.1(iv) implies $H^{f^R}(n, \nabla y_n^{f^R}) = \max_{a \in A(n)} \{\sum_{j \in a} \lambda_j (r_j(n) - \nabla y_n^{f^R})\}$ for $n \in R^f$. From Proposition 3.1(a), $f^R \in C^*$.

Definition 3.1 implies that $T^* \subseteq E^* \subseteq F^*$. In the case with zero holding costs, the result of Feinberg and Reiman [4, Theorem 3.1] implies that $T^* = F^*$. Thus, in the case with zero holding costs, $T^* = C^* = E^* = F^*$. ■

The following example shows that there may exist a canonical policy which is not an ETRP for our problem.

Example 3.2 Consider an M/M/1/5 queue with three customer types, where $\lambda_1 = \lambda_2 = \lambda_3 = \mu = 1$, $r_1(0) = 10$, $r_1(1) = 7$, $r_1(2) = 4$, $r_1(3) = 1$, $r_1(4) = -2$, $r_2(0) = 8$, $r_2(1) = 6$, $r_2(2) = 4$, $r_2(3) = 2$, $r_2(4) = 0$, and $r_3(0) = r_3(1) = r_3(2) = r_3(3) = r_3(4) = 6.5$. Consider a TRP d with $d_0 = \{1, 2, 3\}$, $d_1 = \{1, 3\}$, and $d_2 = d_3 = d_4 = d_5 = \emptyset$. Equations (5) imply that d is a canonical policy with $v^d = 6.5$. Now consider another policy f with $f_0 = \{1, 2, 3\}$, $f_1 = \{1\}$, $f_2 = \{3\}$, and $f_3 = f_4 = f_5 = \emptyset$. Note that $f = f^R$ and f^R is not an ETRP. Equations (5) also imply that f is a canonical policy. ■

The following formula describes an arbitrary canonical policy for Example 3.2:

$$f(n, j) = \begin{cases} \{accept\}, & \text{if } n = 0 \text{ or } (n, j) = (1, 1); \\ \{reject\}, & \text{if either } (n, j) = (1, 2) \text{ or } n = 2, 3, 4 \text{ and } j = 1, 2; \\ arbitrary, & \text{if } n = 1, 2, 3, 4 \text{ and } j = 3. \end{cases}$$

This description contrasts to the result without holding costs [10], where there exist at most two optimal control levels for each customer type. We also observe that, in Example 3.2, there is only one customer type with more than two optimal control levels. In Theorem 3.4, we will show that, if, as stated in Condition 2.1, different customer types have different reward functions, then there exists only one customer type with more than two optimal control levels. In addition, Condition 3.1 provides necessary and sufficient conditions when such customer type exists. In order to formulate Condition 3.1, Definition 3.3, and Theorems 3.3–3.4, we need the following two lemmas.

Lemma 3.3 Let $\tilde{R} = \sum_{j=1}^m \lambda_j \times r_j(0)$. Then $0 < v^* < \tilde{R}$.

Proof: Since $R(n, f_n) \leq \tilde{R}$, $n = 0, 1, \dots, n^f - 1$, $R(n^f, f_{n^f}) = 0$, and $\pi_{n^f}^*(f) > 0$, we have $v^f = \sum_{n=0}^{n^f-1} \pi_n^*(f) R(n, f_n) \leq \sum_{n=0}^{n^f-1} \pi_n^*(f) \tilde{R} < \tilde{R}$. Furthermore, let f accept all customer types if and only if the system is empty. Then $v^* \geq v^f = \pi_0^*(f) \times \tilde{R} > 0$. ■

Equations (5) and Theorem 3.1 imply $\nabla y_{n^f-1}^f = \frac{v^*}{k\mu}$ when $f \in F^*$. For $f \in F^*$, we define $N^f = \min\{n = 0, 1, \dots, n^f - 1 : \nabla y_n^f = \frac{v^f}{k\mu}\}$. This definition yields $N^f \leq n^f - 1$.

Lemma 3.4 If $f, g \in F^*$, then $N^f = N^g$ and $\nabla y_n^f = \nabla y_n^g$ for $n = 0, 1, \dots, \min\{n^f, n^g\} - 1$.

Proof: We prove this lemma by contradiction. Let $n^f = \min\{n^f, n^g\}$. Then state n^f is recurrent under both f and g . Assume that $\nabla y_n^g \neq \nabla y_n^f$ for some $n = 0, 1, \dots, n^f - 1$. Since $g_n \neq \emptyset$ and $f_n \neq \emptyset$, Lemma 3.1 implies $H^g(n, \nabla y_n^g) \neq H^f(n, \nabla y_n^f)$. From (5), $H^g(n, \nabla y_n^g) + \mu_n \nabla y_{n-1}^g = v^* = H^f(n, \nabla y_n^f) + \mu_n \nabla y_{n-1}^f$. Thus, $\nabla y_{n-1}^g \neq \nabla y_{n-1}^f$. By repeating this argument n times, we have $\nabla y_0^g \neq \nabla y_0^f$. Since $f, g \in F^*$, Lemma 3.3 implies $v^f > 0$ and $v^g > 0$. However, since $\nabla y_0^f \neq \nabla y_0^g$, Lemma 3.1 and (5) yield $v^g = H^g(0, \nabla y_0^g) \neq H^f(0, \nabla y_0^f) = v^f$. Thus, either g or f is not optimal. This contradicts our assumption that $f, g \in F^*$. Thus, if $f, g \in F^*$, then $\nabla y_n^g = \nabla y_n^f$ for $n = 0, 1, \dots, n^f - 1$. Since $N^f \leq n^f - 1$ and $n^f = \min\{n^f, n^g\}$, we have $N^g = N^f$. ■

From Lemma 3.4, N^f does not depend on $f \in F^*$. We let $N^* = N^f$ for any optimal policy f . Lemma 3.4 implies

$$N^* \leq \min\{n^f : f \in F^*\} - 1. \quad (6)$$

Example 3.2 satisfies the following condition, which indicates the existence of a pathological customer type i . As shown later, this is a necessary and sufficient condition for the existence of a canonical policy that is not a TRP.

Condition 3.1 $N^* < N - 1$ and there exists a customer type i such that $r_i(n) = \frac{v^*}{k\mu}$ for all $n = 0, 1, \dots, N - 1$.

We need the following definitions before we describe the structures of canonical policies and optimal TRPs.

Definition 3.2 *An integer M_j is called an optimal control level for type j customers if there exists a policy $f \in T^*$ such that $M_j^f = M_j$.*

Definition 3.3 *Let $n_i^f = \min\{n = N^*, N^* + 1, \dots, N : f(n, i) = \{\text{reject}\}\}$. Then a policy $f^{R,i}$ is called the restriction of f for customer type i if $f^{R,i}(n, j) = f(n, j)$ for $j \neq i$, $f^{R,i}(n, i) = \{\text{accept}\}$ for $n < n_i^f$, and $f^{R,i}(n, i) = \{\text{reject}\}$ for $n \geq n_i^f$.*

Theorem 3.3 *If Condition 3.1 does not hold, then: (a) every $f \in F^*$ is an ETRP; (b) $T^* = C^* \subseteq E^* = F^*$; (c) for each customer type $j = 1, 2, \dots, m$, there exists two optimal control levels $M_j, M'_j = 0, 1, \dots, N$ such that*

- (i) $M_l \geq k$ for customer type l with $r_l(0) = \max_{j=1,2,\dots,m} r_j(0)$;
- (ii) $\max_{j=1,2,\dots,m} M_j > N^* \geq k - 1$;
- (iii) either $M'_j = M_j$ or $M'_j = M_j + 1$;
- (iv) a policy f belongs to T^* if and only if it is a TRP with $M_j^f \in \{M_j, M'_j\}$ for all $j = 1, 2, \dots, m$.

In particular, statement (c) in Theorem 3.3 means that, for each customer type j , there exist at most two optimal control levels and, if there are two different optimal control levels M_j and M'_j , where $M'_j > M_j$, then $M'_j = M_j + 1$. A policy is canonical if and only if it is a TRP that follows one of these optimal control levels for each customer type. Theorem 3.3 implies the following corollary.

Corollary 3.1 *If either of the following conditions holds:*

- (i) $r_j(n)$ are decreasing in $n = k - 1, k, \dots, N - 1$;
- (ii) $r_j(n) > r_{j+1}(n)$ for $j = 1, 2, \dots, m - 1$ and $n = 0, 1, \dots, N - 1$;

then statements (a)–(c) of Theorem 3.3 hold.

Figure 1: Condition 3.1 does not hold in Example 3.1

Theorem 3.4 *If Condition 3.1 holds, then, for each customer type $j \in J$, where $J = \{1, 2, \dots, m\} \setminus \{i\}$, there exist two optimal control levels $M_j, M'_j = 0, 1, \dots, N$ such that*

- (i) $M_l \geq k$ for customer type l with $r_l(0) = \max_{j=1,2,\dots,m} r_j(0) > r_i(0)$;
- (ii) $\max_{j \in J} M_j > N^* \geq k - 1$;
- (iii) either $M'_j = M_j$ or $M'_j = M_j + 1$;
- (iv) a policy f belongs to T^* if and only if it is a TRP with $M_j^f \in \{M_j, M'_j\}$, $j \in J$, and $M_i^f \in \{N^*, N^* + 1, \dots, N\}$.

Furthermore, a policy f belongs to C^* if and only if $f^{R,i}$ belongs to T^* .

Statement (iii) in Theorem 3.4 means that, for each customer type j , $j \neq i$, there exist at most two optimal control levels. If there are two different optimal control levels, then one of them is greater than the other by 1. Since $N^* < N - 1$, for customer type i , there exist more than two optimal control levels, which are $N^*, N^* + 1, \dots, N$. A

Figure 2: Condition 3.1 holds in Example 3.2, $i = 3$, and $N^* = 1$

TRP is optimal if and only if it follows these optimal control levels. To prove Theorem 3.3, Theorem 3.4, and Corollary 3.1, we need the following lemmas.

Lemma 3.5 *If $f \in C^*$, then the following statements hold:*

- (i) $H^f(n, \nabla y_n^f) \geq 0$ for $n = 0, 1, \dots, N-1$ and $H^f(n, \nabla y_n^f) = 0$ for $n = n^f, n^f + 1, \dots, N$;
- (ii) if $n^f < N$, then $\nabla y_n^f = \frac{v^*}{k\mu} \geq \max_{j=1,2,\dots,m} r_j(n)$ for $n = n^f, n^f + 1, \dots, N-1$.

Proof: Since $f \in C^*$, Proposition 3.1(b) implies $H^f(n, \nabla y_n^f) \geq 0$ for all $n = 0, 1, \dots, N-1$. If $n^f = N$, then $R^f = \{0, 1, \dots, N\}$. By definition, $H^f(N, \nabla y_N^f) = 0$. Thus, statement (i) holds when $n^f = N$. Now let $n^f \leq N-1$. Since $f \in C^*$ and $f_{n^f} = \emptyset$, Proposition 3.1(c) implies $\nabla y_{n^f}^f \geq \max_{j=1,2,\dots,m} r_j(n^f)$. Theorem 3.1 yields $n^f \geq k$ and (5) implies $\nabla y_{N-1}^f = \nabla y_{n^f-1}^f = \frac{v^*}{k\mu} \geq \max_{j=1,2,\dots,m} r_j(n^f) \geq \max_{j=1,2,\dots,m} r_j(N-1)$. Since $f \in C^*$, Proposition 3.1(b) implies $H^f(N-1, \nabla y_{N-1}^f) = \sum_{j=1}^m \lambda_j(r_j(N-1) - \nabla y_{N-1}^f)^+ = 0$. From (5), $k\mu \nabla y_{N-2}^f = v^*$ and, thus, $\nabla y_{N-2}^f = \frac{v^*}{k\mu}$. By repeating this

argument $N - 1 - n^f$ times, we have, for $n^f \leq n \leq N - 1$,

$$\nabla y_{n^f-1}^f = \nabla y_n^f = \frac{v^*}{k\mu} \geq \max_{j=1,2,\dots,m} r_j(n^f) \geq \max_{j=1,2,\dots,m} r_j(n). \quad (7)$$

Thus, $\nabla y_n^f = \frac{v^*}{k\mu} \geq \max_{j=1,2,\dots,m} r_j(n)$ and $H^f(n, \nabla y_n^f) = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_n^f)^+ = 0$ for $n = n^f, n^f + 1, \dots, N - 1$. $\blacksquare$

Lemma 3.6 *If $f \in C^*$, then $\max_{j=1,2,\dots,m} r_j(0) > \nabla y_n^f$, $n = 0, 1, \dots, N - 1$.*

Proof: We prove this lemma by contradiction. Assume that $r_l(0) = \max_{j=1,2,\dots,m} r_j(0) \leq \nabla y_n^f$ for some $n = 0, 1, \dots, N - 1$. Since $f \in C^*$, from Proposition 3.1(c), $H^f(n, \nabla y_n^f) = 0$. From Lemma 3.5(i), $H^f(n+1, \nabla y_{n+1}^f) \geq 0$. From (5), $H^f(n+1, \nabla y_{n+1}^f) + \mu_{n+1} \nabla y_n^f = v^f = v^* \geq H^f(n+1, \nabla y_{n+1}^f) + \mu_{n+1} r_l(0) \geq \mu_{n+1} r_l(0)$ and $\mu_n \nabla y_{n-1}^f = v^*$ because of $H^f(n, \nabla y_n^f) = 0$. Since $H^f(n+1, \nabla y_{n+1}^f) + \mu_{n+1} \nabla y_n^f = v^* = \mu_n \nabla y_{n-1}^f$ and $H^f(n+1, \nabla y_{n+1}^f) \geq 0$, we have $\mu_n \nabla y_{n-1}^f \geq \mu_{n+1} r_l(0)$. Thus, $\nabla y_{n-1}^f \geq \frac{\mu_{n+1}}{\mu_n} r_l(0) \geq r_l(0)$. By repeating this argument n times, we obtain $\nabla y_0^f \geq r_l(0)$. Since $\nabla y_0^f \geq r_l(0)$, from Proposition 3.1(b) and (5), $H^f(0, \nabla y_0^f) = \sum_{j=1}^m \lambda_j(r_j(0) - \nabla y_0^f)^+ = 0 = v^*$. This contradicts Lemma 3.3. Thus, $r_l(0) > \nabla y_n^f$ for $n = 0, 1, \dots, N - 1$. $\blacksquare$

Lemma 3.7 *The following statements hold:*

$$(i) \text{ if } f \in C^*, \text{ then } k - 1 \leq N^* \text{ and } 0 < \nabla y_0^f < \nabla y_1^f < \dots < \nabla y_{N^*}^f = \nabla y_{N^*+1}^f = \dots = \nabla y_{N-1}^f = \frac{v^*}{k\mu};$$

$$(ii) \text{ if } f, g \in C^*, \text{ then } \nabla y_n^f = \nabla y_n^g \text{ for } n = 0, 1, \dots, N - 1.$$

Proof: See Appendix.

In view of Lemma 3.7 (ii), we set $u_n = \nabla y_n^f$ for any $f \in C^*$. Lemma 3.7(i) implies that

$$0 < u_0 < u_1 < \dots < u_{N^*} = u_{N^*+1} = \dots = u_{N-1} = \frac{v^*}{k\mu}, \quad (8)$$

and this lemma and (6) imply

$$k - 1 \leq N^* \leq \min\{n^f : f \in F^*\} - 1. \quad (9)$$

For $j = 1, 2, \dots, m$, we define M_j and M'_j as follows:

$$M_j = \begin{cases} \min\{n = 0, 1, \dots, N - 1 : r_j(n) \leq u_n\}, & \text{if } r_j(N - 1) \leq u_{N-1}; \\ N, & \text{otherwise.} \end{cases} \quad (10)$$

$$M'_j = \begin{cases} \min\{n = 0, 1, \dots, N - 1 : r_j(n) < u_n\}, & \text{if } r_j(N - 1) < u_{N-1}; \\ N, & \text{otherwise.} \end{cases} \quad (11)$$

Lemma 3.8 *A policy $f \in C^*$ if and only if, for all $j = 1, 2, \dots, m$,*

$$f(n, j) = \begin{cases} \{accept\}, & \text{if } n < M_j; \\ \{reject\}, & \text{if } n \geq M'_j. \end{cases} \quad (12)$$

Proof: Condition 2.1 and (8) imply respectively that the functions $r_j(n)$, $j = 1, 2, \dots, m$, are nonincreasing in n and the numbers u_n are nondecreasing in n . This implies that

$$\{n = 0, 1, \dots, N - 1 : n < M_j\} = \{n = 0, 1, \dots, N - 1 : r_j(n) > u_n\},$$

$$\{n = 0, 1, \dots, N - 1 : M_j \leq n < M'_j\} = \{n = 0, 1, \dots, N - 1 : r_j(n) = u_n\},$$

$$\{n = 0, 1, \dots, N - 1 : M'_j \leq n \leq N - 1\} = \{n = 0, 1, \dots, N - 1 : r_j(n) < u_n\}.$$

Let $f \in C^*$. Then $u_n = \nabla y_n^f$, $n = 0, 1, \dots, N - 1$, and Proposition 3.1(c) implies (12). Now let f satisfy (12). Consider $g \in C^*$. Then, according to the proven necessary part, g satisfies (12) too with f substituted with g in (12). Observe that $H^f(n, u_n) = H^g(n, u_n)$, $n = 0, 1, \dots, N - 1$. From (8), $\nabla y_n^g = u_n$ for all n . Equations (5) imply

$\nabla y_n^f = \nabla y_n^g = u_n$ for all n . Proposition 3.1(c) yields $f \in C^*$. ■

Lemma 3.9 *Consider M_j and M'_j , $j = 1, 2, \dots, m$, defined in (10) and (11). If Condition 3.1 does not hold, then statements (i)–(iv) of Theorem 3.3 hold. If Condition 3.1 holds, then statements (i)–(iv) of Theorem 3.4 hold and, in addition, $M_i = N^*$ and $M'_i = N$.*

Proof: Statement (i) of Theorem 3.3 and the same inequality in statement (i) of Theorem 3.4 follow from Theorem 3.1. If Condition 3.1 holds, then Lemma 3.6 and Lemma 3.7(i) imply that $r_l(0) = \max_{j=1,2,\dots,m} r_j(0) > u_{N-1} = \frac{v^*}{k\mu} = r_i(0)$. Statement (i) of Theorem 3.4 is proven.

For (ii), (5) implies that $H^f(N^*+1, u_{N^*+1}) + k\mu u_{N^*} = v^* = H^f(N^*, u_{N^*}) + k\mu u_{N^*-1}$ for any $f \in C^*$. From (8), $u_{N^*} = \frac{v^*}{k\mu} > u_{N^*-1}$. Therefore, $H^f(N^*+1, u_{N^*+1}) > H^f(N^*, u_{N^*}) \geq 0$, where the second inequality follows from Proposition 3.1(b). Since $H^f(N^*, u_{N^*}) > 0$, Proposition 3.1(b)(c) imply there exists some customer type l , $l \neq i$, such that $r_l(N^*) - u_{N^*} > 0$ and $f(N^*, l) = \{accept\}$ for any $f \in C^*$. Lemma 3.8 implies $M_l > N^*$. For statement (ii) of Theorem 3.3, $\max_{j=1,2,\dots,m} M_j \geq M_l > N^*$. For statement (ii) of Theorem 3.4, since $r_l(N^*) > u_{N^*} = \frac{v^*}{k\mu} = r_i(N^*)$, we have $l \neq i$. Thus, we obtain $\max_{j \in J} M_j \geq M_l > N^*$. The first inequalities in statements (ii) of both theorems are proven. The second inequalities in statements (ii) of both theorems follow from (9).

For (iii), let either Condition 3.1 does not hold or $j \neq i$. If $M_j = N$ or $M_j = N - 1$, then the statement is obvious. Thus, let $M_j < N - 1$. From (10), either $r_j(M_j) = u_{M_j}$ or $r_j(M_j) < u_{M_j}$. If $r_j(M_j) < u_{M_j}$, then, from (10) and (11), we have $M'_j = M_j$. If $r_j(M_j) = u_{M_j}$, then we shall prove that $M'_j = M_j + 1$. Assume that $M'_j > M_j + 1$. Since $M'_j \leq N$, we have $M_j \leq N - 2$. Since $r_j(M_j) = u_{M_j}$ and $M'_j > M_j + 1$, (10) and (11) imply that $r_j(M_j + 1) = u_{M_j+1}$. Since $r_j(M_j + 1) \leq r_j(M_j)$ and $u_{M_j} \leq u_{M_j+1}$, we have $r_j(M_j + 1) = r_j(M_j)$ and $u_{M_j} = u_{M_j+1}$. Formula (8) yields $u_{M_j} = \frac{v^*}{k\mu} = r_j(M_j) = r_j(M_j + 1)$ and $N - 2 \geq M_j \geq N^*$. From (9), $M_j \geq N^* \geq k - 1$. Since

$r_j(M_j) = r_j(M_j + 1) = \frac{v^*}{k\mu}$ and $M_j \geq k - 1$, Condition 2.1 implies that $r_j(n) = \frac{v^*}{k\mu}$ for all $n = 0, 1, \dots, N - 1$ and, thus, $j = i$. From (8) and (10), we have $N^* = M_i \leq N - 2$. Since $N^* < N - 1$ and there exists a customer type i , Condition 3.1 holds. This contradicts our assumption that Condition 3.1 does not hold. Therefore, if either Condition 3.1 does not hold or $j \neq i$, then:

$$M'_j = \begin{cases} M_j + 1, & \text{if and only if } r_j(M_j) = u_{M_j}; \\ M_j, & \text{otherwise.} \end{cases}$$

Statements (iii) of both theorems are proven.

For (iv), we prove statement (iv) of Theorem 3.3 first. Consider $f \in T^*$. Then, from Theorem 3.2, $f = f^R \in C^*$. Since $f \in C^*$, statement (iii) of Theorem 3.3 and Lemma 3.8 imply that $M_j^f \in \{M_j, M'_j\}$ for all $j = 1, 2, \dots, m$. Thus, f is a TRP with $M_j^f \in \{M_j, M'_j\}$ for all $j = 1, 2, \dots, m$. Now let f be a TRP with $M_j^f \in \{M_j, M'_j\}$ for $j = 1, 2, \dots, m$. Lemma 3.8 implies $f \in C^*$. Since $C^* \subseteq F^*$, we have $f \in T^*$. Next we prove statement (iv) of Theorem 3.4. Consider $f \in T^*$. Then, from Theorem 3.2, $f = f^R \in C^*$. Statement (iii) of Theorem 3.4 and Lemma 3.8 imply that $M_j^f \in \{M_j, M'_j\}$, $j \neq i$. In addition, (8) and Condition 3.1 imply that

$$r_i(n) = u_n = \frac{v^*}{k\mu}, \text{ for } n = N^*, N^* + 1, \dots, N - 1, \quad (13)$$

and (10) and (11) imply that $M_i = N^*$ and $M'_i = N$. Lemma 3.8 implies that $M_i^f \in \{N^*, N^* + 1, \dots, N\}$. Now let f be a TRP with $M_j^f \in \{M_j, M'_j\}$, $j \neq i$, and $M_i^f \in \{N^*, N^* + 1, \dots, N\}$. Then Lemma 3.8 implies $f \in C^*$. Since $f \in C^* \subseteq F^*$, $f \in T^*$. ■

Proof of Theorem 3.3: (c) Statments (i)–(iv) follow from Lemma 3.9.

(b) If $f \in C^*$, then statement (iii) of this theorem and Lemma 3.8 imply that $M_j^f \in \{M_j, M'_j\}$ for all $j = 1, 2, \dots, m$ and, thus, $f \in T^*$. Now let $f \in T^*$. Then from Theorem 3.2, $f = f^R \in C^*$. Thus, $T^* = C^*$. Now consider any $f \in F^*$. Then, from

Theorem 3.2, $f^R \in C^*$. Since $C^* = T^*$, $f^R \in T^*$ is a TRP. From Definition 3.1, f is an ETRP and, thus, $f \in E^*$. Since any $f \in F^*$ is an ETRP, we obtain $F^* = E^*$. Since $C^* \subseteq F^*$, we obtain $T^* = C^* \subseteq E^* = F^*$.

(a) The statement follows from (b). ■

Proof of Corollary 3.1: We show that Condition 3.1 does not hold in the either case.

(i) Since $r_j(n)$ are decreasing in $n = k - 1, k, \dots, N - 1$ for all $j = 1, 2, \dots, m$, there does not exist a customer type i with $r_i(n) = \frac{v^*}{k\mu}$ for $n = 0, 1, \dots, N - 1$. Thus, Condition 3.1 does not hold in this case and statements (a)–(c) of Theorem 3.3 hold.

(ii) If $N = k$, then Lemma 3.7(i) implies $N^* = k - 1 = N - 1$. Thus, Condition 3.1 does not hold. Now let $N > k$. If there does not exist a customer type i with $r_i(n) = \frac{v^*}{k\mu}$ for $n = 0, 1, \dots, N - 1$, then Condition 3.1 does not hold. Thus, the remaining case is when $N > k$ and there exists a customer type i with $r_i(n) = \frac{v^*}{k\mu}$ for $n = 0, 1, \dots, N - 1$. In this case, we shall prove that $N^* = N - 1$. Lemma 3.6 states that there exists a customer type l with $r_l(0) = \max_{j=1,2,\dots,m} r_j(0) > \frac{v^*}{k\mu} = r_i(0)$. Since $r_j(n) > r_{j+1}(n)$ for all $n = 0, 1, \dots, N - 1$ and $j = 1, 2, \dots, m - 1$ and $r_l(0) > r_i(0) = \frac{v^*}{k\mu}$, we have that $r_l(n) > r_i(n) = \frac{v^*}{k\mu}$ for all $n = 0, 1, \dots, N - 1$. Equations (5) imply $u_{N-1} = \frac{v^*}{k\mu} < r_l(N - 1)$. Proposition 3.1 implies $f(N - 1, l) = \{\text{accept}\}$ and $H^f(N - 1, u_{N-1}) > 0$. Again, from (5), $H^f(N, u_N) + k\mu u_{N-1} = v^* = H^f(N - 1, u_{N-1}) + k\mu u_{N-2}$. Since $H^f(N - 1, u_{N-1}) > H^f(N, u_N) = 0$, we have $u_{N-1} = \frac{v^*}{k\mu} > u_{N-2}$ and, from Lemma 3.7(i), $N^* = N - 1$. Since $N^* = N - 1$, Condition 3.1 does not hold. Thus, statements (a)–(c) of Theorem 3.3 hold. ■

Proof of Theorem 3.4: Statements (i)–(iv) follow from Lemma 3.9. If $f \in C^*$, then statement (iii) of this theorem and Lemma 3.8 imply that $M_j^f \in \{M_j, M'_j\}$, $j \in J$, and (13), (8), and Lemma 3.8 imply that $M_i = N^*$, $M'_i = N$, and, thus, $n_i^f \geq N^*$. Definition 3.3 implies $M_j^{f^{R,i}} = M_j^f \in \{M_j, M'_j\}$, $j \in J$, and $M_i^{f^{R,i}} = n_i^f \in \{N^*, N^* + 1, \dots, N\}$.

Lemma 3.8 implies $f^{R,i} \in C^*$. Since $f^{R,i} \in C^*$ is a TRP, we have $f^{R,i} \in T^*$. Now let $f^{R,i} \in T^*$. Then $f^{R,i} = f^R \in C^*$. Statements (iii)–(iv) of this theorem and Lemma 3.8 imply that $M_j^{f^{R,i}} \in \{M_j, M'_j\}$, $j \in J$, and $M_i^{f^{R,i}} \in \{N^*, N^* + 1, \dots, N\}$. Since $N^* = M_i \leq M_i^{f^{R,i}}$, from Definition 3.3, $M_j^f = M_j^{f^{R,i}} \in \{M_j, M'_j\}$, $j \in J$, and $f(n, i) = \{accept\}$, $n = 0, 1, \dots, M_i - 1$. Again Lemma 3.8 implies $f \in C^*$ ■

Theorem 3.5 $\emptyset \neq T^* \subseteq C^* \subseteq F^*$.

Proof: According to [20], $C^* \neq \emptyset$. By definition, $C^* \subseteq F^*$. If Condition 3.1 does not hold, then Theorem 3.3(b) implies $T^* = C^* \subseteq F^*$ and $T^* \neq \emptyset$. If Condition 3.1 holds, then Theorem 3.4 implies $\emptyset \neq T^* \subseteq C^*$. Thus, if Condition 3.1 holds, then $T^* \subseteq C^* \subseteq F^*$ and $T^* \neq \emptyset$. ■

Note that Theorem 3.4(ii) states that $\max_{j=1,2,\dots,m} M_j > N^*$. Consider a policy f such that $M_j^f \in \{M_j, M'_j\}$ for $j \neq i$, $f(n, i) = \{accept\}$ for $n = 0, 1, \dots, N^* - 1$, $f(N^*, i) = \{reject\}$, and $f(n, i) = \{accept\}$ for $n = N^* + 1, N^* + 2, \dots, N - 1$. Then f is a canonical policy, but it is not an ETRP. Thus, if Condition 3.1 holds, then C^* is not a subset of E^* .

Remark 3.1 If $\lim_{n \rightarrow \infty} r_j(n) < 0$ for each $j = 1, 2, \dots, m$, then the control admission problem for an infinite capacity queue can be reduced to the problem for a finite capacity queue. Under this natural assumption, for each $j = 1, 2, \dots, m$, there exists a finite number N_j such that $r_j(N_j) < 0$ and it is not optimal to admit customer type j at state n , where $n \geq N_j$. Let $U = \min\{n = k, k + 1, \dots : r_j(n) \leq 0 \text{ for all } j = 1, 2, \dots, m\}$. Then the set C^* for the M/M/k/ ∞ queue is the same as the set C^* for the M/M/k/U queue. In other words, if $h_j(t) > 0$ for all $j = 1, 2, \dots, m$ and $t > 0$, then the control admission problem for M/M/k/ ∞ queues is the same as the problem for M/M/k/U queues.

4 Bias and Blackwell Optimality

If F^* contains two or more policies, bias optimality can be applied to find the policy that maximizes the bias vector among all policies from F^* . Haviv and Puterman [6] considered an M/M/k/N queue with one customer type and convex increasing holding costs. They proved that there exist at most two optimal control levels for the customers and, if there are two different optimal control levels, the difference between them is one. They also showed that the bias optimal policy is unique, is the policy that selects the larger optimal control level for the customers, and is also Blackwell optimal. Lewis et al. [10] extended these results to M/M/k/N queues with multiple customer types and without holding costs. For the problem with several customer types and holding costs, we also show that the bias optimal policy is unique and selects the largest optimal control level for each customer type. Unlike the cases considered in [6] and [10], Example 3.2 demonstrates the possibility that there are more than two optimal control levels for one customer type. For this customer type, the optimal control level under the bias optimal policy is N . Condition 3.1 provides the necessary and sufficient condition when there are more than two optimal control levels for some customer type. This customer type exists if and only if Condition 3.1 holds, this type is the type i described in Condition 3.1, and the number of optimal control levels for any other customer type j cannot exceed two. Below are the definitions of bias optimality and Blackwell optimality.

Definition 4.1 *A policy g is bias optimal if g is optimal and $y_n^g \geq y_n^f$, $n = 0, 1, \dots, N$, and for every policy $f \in F^*$.*

Definition 4.2 *A policy ϕ is Blackwell optimal if there exists a number $\alpha^* > 0$ such that $v_\alpha^\phi \geq v_\alpha^f$ for all $\alpha \in (0, \alpha^*]$ and all $f \in F$, where v_α^f is the total discounted reward under policy f with discount rate α .*

The following theorem describes the bias optimal policy g .

Theorem 4.1 *There exists a unique bias optimal policy g and it is also Blackwell optimal. This policy is a TRP that assigns the largest optimal control level for each customer type. In particular, if Condition 3.1 holds, then g is a TRP with $M_j^g = M_j'$, $j \neq i$, and $M_i^g = N$; if Condition 3.1 does not hold, then g is a TRP with $M_j^g = M_j'$ for $j = 1, 2, \dots, m$.*

Since $y_n^f = y_0^f - \sum_{z=0}^{n-1} \nabla y_z^f$, $n = 1, 2, \dots, N$, equation (4) can be rewritten as

$$\pi_0^*(f)y_0^f + \sum_{z=1}^N \pi_z^*(f)(y_0^f - \sum_{n=0}^{z-1} \nabla y_n^f) = 0.$$

From $\sum_{z=0}^N \pi_z^*(f) = 1$, we have

$$y_0^f = \sum_{z=1}^N \pi_z^*(f) \sum_{n=0}^{z-1} \nabla y_n^f. \quad (14)$$

In addition, $y_n^f = y_0^f - \sum_{z=0}^{n-1} \nabla y_z^f$, $n = 1, 2, \dots, N$.

For discrete-time MDPs, Veinott [17] proved the existence of bias optimal policies and provided an algorithm for their computation. It is also well-known for discrete-time MDPs with finite state and action sets that bias optimal policies are canonical. In particular, one of the most popular versions of Howard's policy iteration algorithm generates a finite sequence of policies $\{f^n\}$ with the property that the pairs (v^{f^n}, y^{f^n}) lexicographically increase, the algorithm stops when the pair cannot be lexicographically improved by following the rules of the algorithm, and the last generated policy is canonical [14, p.461]. Since a bias optimal policy cannot be improved by the algorithm, it is canonical. This is also true for CTMDPs, because uniformization [14] preserves the average rewards, biases, and canonical equations. Thus, if g is a bias optimal policy for our problem, then $g \in C^*$.

Lemma 4.1 *If g is a bias optimal policy and $r_j(n) = u_n$ for some $n = 0, 1, \dots, N-1$ and some $j = 1, 2, \dots, m$, then $g(n, j) = \{\text{accept}\}$.*

Proof: The proof is based on contradiction. Let g be a bias optimal policy and $g(n, j) = \{reject\}$ at a pair (n, j) such that $n = 0, 1, \dots, N-1$, $j = 1, 2, \dots, m$, and $r_j(n) = u_n$. Consider two cases: (i) $n \in R^g$ and (ii) $n \in T^g$.

(i) Let $n \in R^g$. This means $\pi_n^*(g) > 0$. For any policy d , let $X_\infty(d) \sim \pi^*(d)$ be the limiting number of customers in the queue. Consider the policy f coinciding with g everywhere, except at (n, j) . Of course, $f(n, j) = \{accept\}$. Since $g \in C^*$, Lemma 3.8 implies that $f \in C^*$. Observe that $\lambda_s^g = \lambda_s^f$ for $s \neq n$ and $\lambda_n^f = \lambda_n^g + \lambda_j$.

Since $\lambda_s^f \geq \lambda_s^g$ for all $s = 0, \dots, N-1$, the well-known comparison results for birth-end-death processes, see, e.g., [12, Theorem 5.2.21], imply that $X_\infty(g) \leq_{st} X_\infty(f)$, where $\leq_{st}$ is the usual stochastic order. In particular, for any increasing numbers $c_0 < c_1 < \dots < c_N$

$$\sum_{s=0}^N \pi_s^*(g) c_n \leq \sum_{s=0}^N \pi_s^*(f) c_n, \quad (15)$$

and the equality holds if and only if $\pi^*(g) = \pi^*(f)$. Since $\lambda_n^g \pi_n^*(g) = \mu_{n+1} \pi_{n+1}^*(g)$, $(\lambda_n^g + \lambda_j) \pi_n^*(f) = \mu_{n+1} \pi_{n+1}^*(f)$, and $\pi_n^*(g) > 0$, the equality $\pi^*(g) = \pi^*(f)$ is impossible. Thus, the strong inequality holds in (15). Apply this inequality to $c_0 = 0$ and $c_s = \sum_{z=0}^{s-1} u_z$, $s = 1, 2, \dots, N$, and recall that $u_s = \nabla y_s^f = \nabla y_s^g$ for all $s = 0, 1, \dots, N-1$, since $f, g \in C^*$. Formula (14) implies

$$y_0^g = \sum_{z=1}^N \pi_z^*(g) \sum_{s=0}^{z-1} u_s < \sum_{z=1}^N \pi_z^*(f) \sum_{s=0}^{z-1} u_s = y_0^f.$$

Thus, g is not bias optimal. This contradiction implies that case (i) is impossible.

(ii) Let $n \in T^g$. Then $n^g < n \leq N-1$. From (6) we have $N^* < n^g - 1 < N-1$. From Lemma 3.5(ii), $u_s = \frac{v^*}{k\mu} \geq \max_{l=1,2,\dots,m} r_l(s)$ for $s = n^g, n^g + 1, \dots, N-1$. Thus, $r_j(n^g) \leq \frac{v^*}{k\mu}$. By Condition 2.1, $r_j(n^g) \geq r_j(n) = u_n = \frac{v^*}{k\mu}$. So, $r_j(n^g) = r_j(n)$. Condition 2.1 implies that $r_j(s)$ is constant: $r_j(s) = \frac{v^*}{k\mu}$ for all $s = 0, \dots, N-1$. Thus, Condition 3.1 holds and $j = i$.

Observe that $g(n^g, j) = \{reject\}$ and $n^g \in R^g$. According to case (i), $y_0^g < y_0^f$ for the policy f that coincides with g at all pairs (n', j') except the pair (n^g, j) , where $f(n^g, j) = \{accept\}$. Thus, g is not bias optimal. This contradiction completes the proof of the lemma. ■

Proof of Theorem 4.1: Lemmas 3.8 and 4.1 imply that a policy g is bias optimal if and only if for all $j = 1, 2, \dots, m$

$$f(n, j) = \begin{cases} \{accept\}, & \text{if } n < M'_j; \\ \{reject\}, & \text{if } n \geq M'_j. \end{cases}$$

Since the bias optimal policy is unique, it is Blackwell optimal, because Blackwell optimal policies are bias optimal. For discrete-time MDPs, this fact is in [14, Thm. 10.1.5], and for CTMDPs this follows from the discrete-time result and uniformization. The remaining statements follow from Theorems 3.3 and 3.4. ■

From Theorem 4.1, we know that the bias optimal policy g is the policy that selects the largest optimal control level for each customer type. Thus, after obtaining a canonical policy f from policy iteration algorithm, the bias optimal policy g can be found by the following formula:

$$g_n = f_n \bigcup \{j = 1, 2, \dots, m : r_j(n) = u_n\} \quad \text{for } n = 0, 1, \dots, N-1.$$

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Appendix: Proofs of Lemmas 3.1 and 3.7

Proof of Lemma 3.1: For $f, d \in F$, let $e_n = H^d(n, \nabla y_n^f) - H^f(n, \nabla y_n^f)$. Then $e_n = H^d(n, \nabla y_n^d) - H^f(n, \nabla y_n^f) + \lambda_n^d(\nabla y_n^d - \nabla y_n^f)$. Equations (5) imply $v^d - v^f =$

$e_n + \mu_n(\nabla y_{n-1}^d - \nabla y_{n-1}^f) - \lambda_n^d(\nabla y_n^d - \nabla y_n^f)$. By multiplying both sides with $\pi_n^*(d)$, summing them in n and using the fact that $\lambda_n^d \pi_n^*(d) = \mu_{n+1} \pi_{n+1}^*(d)$, we have

$$\sum_{n=0}^N \pi_n^*(d)(v^d - v^f) = v^d - v^f = \sum_{n=0}^N \pi_n^*(d)e_n. \quad (16)$$

(16) is a particular form of formula (8.39) in [5] and we use it to prove (i)–(iii).

(i) Assume that $r_j(n) > \nabla y_n^f$ and $f(n, j) = \{reject\}$ for some $n \in R^f$. Then consider a policy g such that $g_s = f_s$ for $s \neq n$, and $g_n = f_n \cup \{j\}$. Observe that $n \in R^g$ and, therefore, $\pi_n^*(g) > 0$. Then, from (16)

$$v^g - v^f = \pi_n^*(g)\lambda_j(r_j(n) - \nabla y_n^f).$$

Since $\pi_n^*(g) > 0$, $\lambda_j > 0$, and $r_j(n) - \nabla y_n^f > 0$, we have $v^g - v^f > 0$. Thus, $f \notin F^*$.

(ii) Assume that $r_j(n) < \nabla y_n^f$ and $f(n, j) = \{accept\}$ for some $n \in R^f$. Then consider a policy g such that $g_s = f_s$ for $s \neq n$, and $g_n = f_n \setminus \{j\}$. Observe that $n \in R^g$. Then, from (16)

$$v^g - v^f = -\pi_n^*(g)\lambda_j(r_j(n) - \nabla y_n^f).$$

Since $\pi_n^*(g) > 0$, $\lambda_j > 0$, and $r_j(n) - \nabla y_n^f < 0$, we have $v^g - v^f > 0$ and, thus, $f \notin F^*$.

(iii) For $n = 0, 1, \dots, N-1$, let $Z_n = \{j = 1, 2, \dots, m : r_j(n) = \nabla y_n^f\}$. Consider any policy $g \in F$ such that $g_n \setminus Z_n = f_n \setminus Z_n$ for all $n = 0, 1, \dots, N-1$. Then $H^g(n, \nabla y_n^f) - H^f(n, \nabla y_n^f) = 0$ for all $n = 0, 1, \dots, N-1$, and (16) implies $v^g - v^f = 0$. Thus, if $f \in F^*$, then $g \in F^*$. Since $v^g = v^f = v^*$, from (5), $\nabla y_{N-1}^g = \nabla y_{N-1}^f = \frac{v^*}{k\mu}$. Since $g(n, j) = f(n, j)$ when $r_j(n) \neq \nabla y_n^f$, by solving (5) with g and f , we have that $\nabla y_{N-2}^g = \nabla y_{N-2}^f$. By repeating this argument $N-1$ times, we have $\nabla y_n^g = \nabla y_n^f$ for $n = 0, 1, \dots, N-1$.

(iv) From (i)–(iii) of this lemma, $\max_{a \in A(n)} \{\sum_{j \in a} \lambda_j(r_j(n) - \nabla y_n^f)\} = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_n^f)^+ = \sum_{j \in f_n} \lambda_j(r_j(n) - \nabla y_n^f) = H^f(n, \nabla y_n^f) \geq 0$ for $n = 0, 1, \dots, n^f$. $\blacksquare$

Proof of Lemma 3.7: (i) If $\nabla y_0^f \leq 0$, then $v^* = H^f(0, \nabla y_0^f) = \sum_{j=1}^m \lambda_j(r_j(n) - \nabla y_0^f) \geq \sum_{j=1}^m \lambda_j r_j(0)$, where the first equality follows from (5) and the second equality follows from Proposition 3.1(b). Therefore, $v^* \geq \sum_{j=1}^m \lambda_j r_j(0)$ and this contradicts Lemma 3.3. Thus, $\nabla y_0^f > 0$. Next we show that $\nabla y_n^f < \nabla y_{n+1}^f$ for $n = 0, 1, \dots, k-2$. From Theorem 3.1, $f_n \neq \emptyset$ for $n = 0, 1, \dots, k-1$. Assume $\nabla y_n^f \geq \nabla y_{n+1}^f$ for some n , $0 \leq n \leq k-2$. Since $f_n \neq \emptyset$, $f_{n+1} \neq \emptyset$, $\nabla y_n^f \geq \nabla y_{n+1}^f$, and $r_j(n) = r_j(n+1) = r_j(0)$ for $j = 1, 2, \dots, m$, from Proposition 3.1(b), we have $H^f(n, \nabla y_n^f) = \sum_{j=1}^m \lambda_j(r_j(0) - \nabla y_n^f)^+ \leq \sum_{j=1}^m \lambda_j(r_j(0) - \nabla y_{n+1}^f)^+ = H^f(n+1, \nabla y_{n+1}^f)$. From (5), $H^f(n+1, \nabla y_{n+1}^f) + (n+1)\mu \nabla y_n^f = v^* = H^f(n, \nabla y_n^f) + n\mu \nabla y_{n-1}^f$. Since $H^f(n, \nabla y_n^f) \leq H^f(n+1, \nabla y_{n+1}^f)$, we have $\nabla y_{n-1}^f \geq \frac{n}{n-1} \nabla y_n^f > \nabla y_n^f$. By repeating this argument n times, we have

- a. $\nabla y_{n+1}^f \leq \nabla y_n^f < \dots < \nabla y_1^f < \nabla y_0^f$;
- b. $H^f(0, \nabla y_0^f) < H^f(1, \nabla y_1^f) < \dots < H^f(n, \nabla y_n^f) \leq H^f(n+1, \nabla y_{n+1}^f)$.

From (5), $H^f(0, \nabla y_0^f) = v^* = H^f(1, \nabla y_1^f) + \mu_1 \nabla y_0^f$. Since $H^f(0, \nabla y_0^f) \leq H^f(1, \nabla y_1^f)$, we have $\nabla y_0^f \leq 0$. This contradicts the result that $\nabla y_0^f > 0$. Therefore, if $f \in C^*$, then

$$0 < \nabla y_0^f < \nabla y_1^f < \dots < \nabla y_{k-1}^f. \quad (17)$$

From Theorem 3.1, $n^f \geq k$ for any $f \in C^*$. If $n^f = k$, then (7) implies that $\nabla y_{k-1}^f = \nabla y_k^f = \dots = \nabla y_{N-1}^f = \frac{v^*}{k\mu}$, $N^* = k-1$, and (i) is proven. Thus, in the rest of the proof of (i), we consider $n^f > k$.

Let $n = k-1, k, \dots, n^f - 2$. First we shall prove that $\nabla y_n^f \leq \nabla y_{n+1}^f$. Assume $\nabla y_n^f > \nabla y_{n+1}^f$ for some $n = k-1, k, \dots, n^f - 2$. From (5), $H^f(n+1, \nabla y_{n+1}^f) + k\mu \nabla y_n^f = v^* = H^f(n+2, \nabla y_{n+2}^f) + k\mu \nabla y_{n+1}^f$. Thus, $H^f(n+1, \nabla y_{n+1}^f) < H^f(n+2, \nabla y_{n+2}^f)$. Since $H^f(n+1, \nabla y_{n+1}^f) < H^f(n+2, \nabla y_{n+2}^f)$, Proposition 3.1(b) implies $H^f(n+1, \nabla y_{n+1}^f) = \sum_{j=1}^m \lambda_j(r_j(n+1) - \nabla y_{n+1}^f)^+ < \sum_{j=1}^m \lambda_j(r_j(n+2) - \nabla y_{n+2}^f)^+ = H^f(n+2, \nabla y_{n+2}^f)$. Since $r_j(n+1) \geq r_j(n+2)$ for $j = 1, 2, \dots, m$, we have $\nabla y_{n+1}^f > \nabla y_{n+2}^f$. By repeating this argument $n^f - 1 - n$ times, we have

- a. $\nabla y_{n^f}^f < \nabla y_{n^f-1}^f < \cdots < \nabla y_{n+1}^f < \nabla y_n^f$;
- b. $H^f(n^f, \nabla y_{n^f}^f) > H^f(n^f - 1, \nabla y_{n^f-1}^f) > \cdots > H^f(n + 2, \nabla y_{n+2}^f) > H^f(n + 1, \nabla y_{n+1}^f)$.

Since $f_{n^f} = \emptyset$, we have $H^f(n^f, \nabla y_{n^f}^f) = 0$. Thus, $0 > H^f(n^f - 1, \nabla y_{n^f-1}^f)$. Lemma 3.5(i) implies $f \notin C^*$ and this contradicts $f \in C^*$. Therefore, if $f \in C^*$, then

$$\nabla y_{k-1}^f \leq \nabla y_k^f \leq \cdots \leq \nabla y_{n^f-1}^f. \quad (18)$$

From (7), (17), (18), $\nabla y_{k-2}^f < \nabla y_{k-1}^f \leq \cdots \leq \nabla y_{n^f-1}^f = \nabla y_{n^f}^f = \cdots = \nabla y_{N-1}^f = \frac{v^*}{k\mu}$. By definition, we obtain $k - 1 \leq N^* \leq n^f - 1$ and

$$\nabla y_{k-1}^f \leq \cdots \leq \nabla y_{N^*-1}^f < \nabla y_{N^*}^f = \nabla y_{N^*+1}^f = \cdots = \nabla y_{N-1}^f = \frac{v^*}{k\mu}. \quad (19)$$

Next we prove that $\nabla y_n < \nabla y_{n+1}$ for $n = k - 1, k, \dots, N^* - 2$. Assume that $\nabla y_n^f = \nabla y_{n+1}^f$ for some $k - 1 \leq n \leq N^* - 2$. Then (5) yields $H^f(n + 1, \nabla y_{n+1}^f) = H^f(n + 2, \nabla y_{n+2}^f)$. If $\nabla y_{n+1}^f < \nabla y_{n+2}^f$, then Proposition 3.1(b) implies that the only possibility is that $H^f(n + 1, \nabla y_{n+1}^f) = H^f(n + 2, \nabla y_{n+2}^f) = 0$. From (5), $H^f(n + 2, \nabla y_{n+2}^f) + k\mu \nabla y_{n+1}^f = H^f(n + 3, \nabla y_{n+3}^f) + k\mu \nabla y_{n+2}^f$. Since $\nabla y_{n+1}^f < \nabla y_{n+2}^f$ and $H^f(n + 2, \nabla y_{n+2}^f) = 0$, we have $H^f(n + 3, \nabla y_{n+3}^f) < 0$ and this contradicts Proposition 3.1(b). Thus, $\nabla y_{n+1}^f = \nabla y_{n+2}^f$. By repeating this argument $N^* - n$ times, we have $\nabla y_n^f = \nabla y_{n+1}^f = \cdots = \nabla y_{N^*-1}^f = \nabla y_{N^*}^f = \frac{v^*}{k\mu}$ and this contradicts (19). Therefore, $\nabla y_n^f < \nabla y_{n+1}^f$ for $n = k - 1, k, \dots, N^* - 2$ and, from (17) and (19), (i) is proven.

(ii) Lemma 3.4 implies that $\nabla y_n^g = \nabla y_n^f$ for all $n = 0, 1, \dots, N^*$. From Lemma 3.7(i), $\nabla y_{N^*}^f = \nabla y_{N^*+1}^f = \cdots = \nabla y_{N-1}^f = \frac{v^*}{k\mu}$ for any $f \in C^*$. Thus, if $f, g \in C^*$, then $\nabla y_n^f = \nabla y_n^g$ for all $n = 0, 1, \dots, N - 1$. ■