ON ASYMPTOTIC OPTIMALITY OF THE SECOND ORDER IN THE MINIMAX QUICKEST DETECTION PROBLEM OF DRIFT CHANGE FOR BROWNIAN MOTION*

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Abstract. This paper deals with the minimax quickest detection problem of a drift change for the Brownian motion. The following minimax risks are studied: $C(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_{\theta} \mathbf{E}_{\theta}(\tau - \theta | \tau \geq \theta)$ and $\overline{C}(T) = \inf_{\overline{\tau} \in \overline{\mathfrak{M}}_T} \sup_{\theta} \mathbf{E}_{\theta}(\overline{\tau} - \theta | \overline{\tau} \geq \theta)$, where $\mathfrak{M}_T$ is the set of stopping times τ such that $\mathbf{E}_{\infty}\tau = T$ and $\overline{\mathfrak{M}}_T$ is the set of randomized stopping times $\overline{\tau}$ such that $\mathbf{E}_{\infty}\overline{\tau} = T$. The goal of this paper is to obtain for these risks estimates from above and from below. Using these estimates we prove the existence of stopping times, which are asymptotically optimal of the first and second orders as $T \rightarrow \infty$ (for $C(T)$ and $\overline{C}(T)$, respectively).

Key words. disorder problem, Brownian motion, minimax risk, asymptotical optimality of the first and second orders

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1. Introduction. Problem formulation and discussion of results.

1.1. The Bayesian formulation of the quickest detection problem of drift change for Brownian motion (“disorder problem”) was given in [6], [7]. In these papers the solution of this problem was given for the case of the exponential a priori distribution of the disorder time and using a special *passage to the limit*, a solution of the disorder problem in the generalized Bayesian setting was obtained (the time when the “disorder” appears is interpreted as a generalized random variable with the “uniform” distribution on $\mathbf{R}^+ = [0, \infty)$). Let us remark that the problem of the quickest detection of a disturbance of a stationary regime is reduced to the generalized Bayesian disorder problem (see [7], [21, subsection 4.2], [22, section 3.9]).

The direct solution of the generalized Bayesian problem without passage to the limit was obtained in [11] (Variant (B)). In [11, Theorem 3.1] estimates from above and from below for *minimax* risk were also obtained. The minimax risk characterizes the detection delay of the disorder time (Variant (C)).

In order to decrease the “gap” between the upper and lower bounds for the minimax risk (see inequalities (13) below and inequalities (3.1) in [11]), in the present paper we consider (for giving an alarm signal that the disorder has appeared) not only Markov times (stopping times), which are defined by the history of the observed process, but *randomized* stopping times as well (Variant ($\overline{C}$)).

The main result of this work is the following. We prove that in the class of the randomized stopping times there exists the stopping time, which is asymptotically

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optimal of the *second order* as $T \rightarrow \infty$; see (14) in subsection 1.6. (For this randomized stopping time the estimates from above and below of the respective minimax risk give a smaller “gap” than for nonrandomized stopping times. Moreover, the first *two* terms in these estimates coincide and the other terms tend to zero as $T \rightarrow \infty$.)

1.2. The statement of the disorder problems in Variants (B), $(\bar{B})$, (C), and $(\bar{C})$ is the following (cf. with [11]).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space, and let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion ($\mathbf{E}B_t = 0$, $\mathbf{E}B_t^2 = t$).

Assume that the observed process $X = (X_t)_{t \geq 0}$ has the following structure:

$$(1) \quad X_t = \mu(t - \theta)^+ + \sigma B_t,$$

where $x^+ = \max(x, 0)$, $\mu \neq 0$, $\sigma > 0$, and θ is the parameter taking values in $[0, \infty]$ and interpreted as the time, when the *disorder* appears. It follows from (1) that

$$dX_t = \begin{cases} \sigma dB_t, & t < \theta, \\ \mu dt + \sigma dB_t, & t \geq \theta, \end{cases}$$

i.e., at the time θ the drift of the Brownian motion changes (i.e., “disorder” happens).

Let us denote by $\mathbf{P}_\theta = \text{Law}(X | \mathbf{P}, \theta)$ the probability distribution (in the space of continuous functions $C = C[0, \infty)$ with Borel σ -field $\mathcal{C}$) of the process X under the assumption that the disorder happened at time θ . In this case the distribution $\mathbf{P}_\infty$ is the probability distribution of the process X , when $\theta = \infty$ (this means that $\mathbf{P}_\infty = \text{Law}(\sigma B_t, t \geq 0)$); the distribution $\mathbf{P}_0$ is the probability distribution of the process X under the assumption that the disorder happened at time $\theta = 0$ (this means that $\mathbf{P}_0 = \text{Law}(\mu t + \sigma B_t, t \geq 0)$).

In order to simplify the notation we suppose (without loss of generality) that in the initial filtered probability space the filtration $(\mathcal{F}_t)_{t \geq 0}$ coincides with $(\mathcal{F}_t^B)_{t \geq 0}$, where $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$ are sigma-fields, generated by the Brownian motion. Obviously, $(\mathcal{F}_t^B)_{t \geq 0} = (\mathcal{F}_t^X)_{t \geq 0}$ for any $\theta \in [0, \infty]$.

Let $\mathfrak{M} = \{\tau: \tau = \tau(\omega) < \infty, \omega \in \Omega\}$ be a class of finite Markov times (stopping times) with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ ($= (\mathcal{F}_t^B)_{t \geq 0}$).

Besides the class $\mathfrak{M}$ (of nonrandomized stopping times) we introduce a larger class of *randomized* stopping times $\bar{\tau} = \bar{\tau}(\bar{\omega})$ defined as follows (cf. [8, Chap. II, section 7]).

Let $\tilde{\Omega} = [0, 1]$, $\tilde{\mathcal{F}} = \mathcal{B}([0, 1])$ be a Borel system of sets on $\tilde{\Omega}$, and let $\tilde{\mathbf{P}} = \lambda$ be a Lebesgue measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$. Let us introduce a new filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t \geq 0}, \bar{\mathbf{P}})$ with $\bar{\Omega} = \Omega \times \tilde{\Omega}$, $\bar{\mathcal{F}} = \mathcal{F} \otimes \tilde{\mathcal{F}}$, $\bar{\mathcal{F}}_t = \mathcal{F}_t \otimes \tilde{\mathcal{F}}$, $\bar{\mathbf{P}} = \mathbf{P} \otimes \tilde{\mathbf{P}}$.

Set $\bar{\omega} = (\omega, \tilde{\omega})$. We shall say that times $\bar{\tau} = \bar{\tau}(\bar{\omega})$ are *randomized* Markov times (stopping times) if $\{\bar{\omega}: \bar{\tau}(\bar{\omega}) \leq t\} \in \bar{\mathcal{F}}_t$ for every $t \geq 0$.

We denote the class of such finite randomized stopping times by $\bar{\mathfrak{M}} = \{\bar{\tau}: \bar{\tau} = \bar{\tau}(\bar{\omega}) < \infty, \bar{\omega} \in \bar{\Omega}\}$.

Now we introduce the following notation. If $Y = Y(\omega)$ and $\tilde{Y} = \tilde{Y}(\tilde{\omega})$ are random elements defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, respectively, then they can naturally be considered as elements on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ if we put $\bar{Y}(\bar{\omega}) = Y(\omega)$ and $\tilde{\bar{Y}}(\bar{\omega}) = \tilde{Y}(\tilde{\omega})$ for $\bar{\omega} = (\omega, \tilde{\omega})$.

We interpret stopping times τ and $\bar{\tau}$ as the times (randomized and nonrandomized) when the “disorder” alarms appeared.

1.3. Let for any $T > 0$

$$(2) \quad \mathfrak{M}_T = \{\tau \in \mathfrak{M}: \mathbf{E}_\infty \tau = T\}, \quad \bar{\mathfrak{M}}_T = \{\bar{\tau} \in \bar{\mathfrak{M}}: \bar{\mathbf{E}}_\infty \bar{\tau} = T\},$$

where $\mathbf{E}_\infty$ and $\overline{\mathbf{E}}_\infty$ are expectations with respect to the measures $\mathbf{P}_\infty$ and $\overline{\mathbf{P}}_\infty = \mathbf{P}_\infty \otimes \overline{\mathbf{P}}$, respectively.

Actually $\mathfrak{M}_T$ is the class of stopping times $\tau \in \mathfrak{M}$ for which the mean time $\mathbf{E}_\infty \tau$ until the disorder alarm signal (though there is no disorder) is equal to T . In other words, $\mathbf{E}_\infty \tau$ is the mean time until the “false disorder alarm.”

For $T > 0$, let us introduce the following quantities, which characterize the quality of stopping times of the classes $\mathfrak{M}_T$ and $\overline{\mathfrak{M}}_T$:

$$B(T) = \inf_{\tau \in \mathfrak{M}_T} \frac{1}{T} \int_0^\infty \mathbf{E}_\theta(\tau - \theta)^+ d\theta, \quad \overline{B}(T) = \inf_{\overline{\tau} \in \overline{\mathfrak{M}}_T} \frac{1}{T} \int_0^\infty \overline{\mathbf{E}}_\theta(\overline{\tau} - \theta)^+ d\theta,$$

where $x^+ = \max(x, 0)$, and

$$(3) \quad C(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_{\theta \geq 0} \mathbf{E}_\theta(\tau - \theta | \tau \geq \theta), \quad \overline{C}(T) = \inf_{\overline{\tau} \in \overline{\mathfrak{M}}_T} \sup_{\theta \geq 0} \overline{\mathbf{E}}_\theta(\overline{\tau} - \theta | \overline{\tau} \geq \theta).$$

We can interpret $\int_0^\infty \mathbf{E}_\theta(\tau - \theta)^+ d\theta$ and $\int_0^\infty \overline{\mathbf{E}}_\theta(\overline{\tau} - \theta)^+ d\theta$ as (generalized) Bayesian risks for stopping times τ and $\overline{\tau}$. They characterize disorder detection delay, when the “uniform” distribution on $[0, \infty)$ is used as an a priori distribution for θ (as the “least favorable” distribution).

Variants (B) and $(\overline{B})$ of the disorder problem consist of finding $B(T)$ and $\overline{B}(T)$, respectively, as well as the corresponding optimal stopping times, if they exist.

The values $C(T)$ and $\overline{C}(T)$ defined in (3) are minimax risks. *Variants (C) and $(\overline{C})$* of the disorder problem consist of finding $C(T)$ and $\overline{C}(T)$, respectively, as well as the corresponding optimal or ε -optimal stopping times, if they exist.

1.4. Let us briefly describe the results which were obtained in [11] for Variants (B) and (C). This will help us to understand why we should use randomized stopping times (in Variants $(\overline{B})$ and $(\overline{C})$).

The process $\psi = (\psi_t)_{t \geq 0}$ plays the key role for the solution of the disorder problem in Variant (B) and is constructed on the basis of the observed process $X = (X_t)_{t \geq 0}$ as a solution of the stochastic differential equation

$$(4) \quad d\psi_t = dt + \frac{\mu}{\sigma^2} \psi_t dX_t, \quad \psi_0 = 0.$$

The process $\psi = (\psi_t)_{t \geq 0}$ can be represented by the formula

$$(5) \quad \psi_t = \int_0^t \frac{L_t}{L_\theta} d\theta, \quad t \geq 0,$$

where

$$L_t = \frac{d(\mathbf{P}_0 | \mathcal{F}_t^X)}{d(\mathbf{P}_\infty | \mathcal{F}_t^X)}, \quad \mathcal{F}_t^X = \sigma(X_s, s \leq t).$$

It is well known (see, for example, [4]) that in model (1)

$$L_t = \exp \left\{ \frac{\mu}{\sigma^2} X_t - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right\}.$$

By Itô's formula, we obtain that

$$dL_t = \frac{\mu}{\sigma^2} L_t dX_t, \quad L_0 = 1.$$

The fact that the process $(\psi_t)_{t \geq 0}$ defined by (5) is a solution of (4) can be checked by application of Itô's formula to the product $L_t \int_0^t (1/L_\theta) d\theta$, $t \geq 0$.

In what follows we will consider μ and σ such that the "signal-to-noise ratio" $\rho = \mu^2/(2\sigma^2)$ is equal to 1.

This assumption simplifies the subsequent relations and at the same time there is no loss of generality since all the characteristics $(B(T), \overline{B}(T), \dots)$ under consideration depend on μ and σ only via the "signal-to-noise ratio" ρ . Conversion of formulas from the case $\rho = 1$ to the general case of the arbitrary values of $\rho > 0$ is straightforward (see [11]).

The following proposition sums up the results obtained in [11] for Variant (B).

PROPOSITION 1.

I. For any $T > 0$

(a) the equality

$$(6) \quad B(T) = \inf_{\tau \in \mathfrak{M}_T} \frac{1}{T} \mathbf{E}_\infty \int_0^\tau \psi_t dt$$

holds;

(b) $\inf_{\tau \in \mathfrak{M}_T}$ in (6) is attained for the stopping times

$$\tau_T^* = \inf\{t \geq 0: \psi_t \geq T\}$$

(and, thereby, this stopping time is optimal in the class $\mathfrak{M}_T$ in Variant (B));

(c) the equality

$$(7) \quad B(T) = F(T^{-1}) - \Delta(T^{-1})$$

holds, where for any $b > 0$

$$(8) \quad F(b) = e^b (-\text{Ei}(-b)), \quad \Delta(b) = 1 - b \int_0^\infty e^{-bu} \frac{\log(1+u)}{u} du,$$

and $-\text{Ei}(-b) = \int_b^\infty t^{-1} e^{-t} dt$ is the integral exponential function.

II. For small and large T

$$(9) \quad B(T) = \begin{cases} \frac{T}{2} + O(T^2), & T \rightarrow 0, \\ \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T), & T \rightarrow \infty, \end{cases}$$

where $\mathbb{C} = 0.577 \dots$ is Euler's constant.

1.5. In order to describe the results in Variant (C), first let us notice that if $\tau \in \mathfrak{M}_T$, then

$$(10) \quad \begin{aligned} \frac{1}{T} \int_0^\infty \mathbf{E}_\theta(\tau - \theta)^+ d\theta &= \frac{1}{T} \int_0^\infty \mathbf{E}_\theta(\tau - \theta \mid \tau \geq \theta) \mathbf{P}_\theta\{\tau \geq \theta\} d\theta \\ &= \frac{1}{T} \int_0^\infty \mathbf{E}_\theta(\tau - \theta \mid \tau \geq \theta) \mathbf{P}_\infty\{\tau \geq \theta\} d\theta, \end{aligned}$$

as $\{\tau \geq \theta\} \in \mathcal{F}_\theta^X$, and for an event $A \in \mathcal{F}_\theta^X$ and any $\theta > 0$ the probability $\mathbf{P}_\theta(A)$ is equal to the probability $\mathbf{P}_\infty(A)$.

Using (10) and the equality $T^{-1} \int_0^\infty \mathbf{P}_\infty\{\tau \geq \theta\} d\theta = T^{-1} \mathbf{E}_\infty \tau = 1$, we obtain that

$$(11) \quad \begin{aligned} \frac{1}{T} \int_0^\infty \mathbf{E}_\theta(\tau - \theta)^+ d\theta &\leq \frac{1}{T} \int_0^\infty \left\{ \sup_{\theta \geq 0} \mathbf{E}_\theta(\tau - \theta \mid \tau \geq \theta) \right\} \mathbf{P}_\infty(\tau \geq \theta) d\theta \\ &= \sup_{\theta \geq 0} \mathbf{E}_\theta(\tau - \theta \mid \tau \geq \theta). \end{aligned}$$

Therefore, $\inf_{\tau \in \mathfrak{M}_T} T^{-1} \int_0^\infty \mathbf{E}_\theta(\tau - \theta)^+ d\theta \leq \inf_{\tau \in \mathfrak{M}_T} \sup_{\theta} \mathbf{E}_\theta(\tau - \theta \mid \tau \geq \theta)$, i.e.,

$$(12) \quad B(T) \leq C(T).$$

Thus we have the estimate of $C(T)$ from below. Combining this estimate with the estimate from above $C(T) \leq F(T^{-1})$ ($= C^*(T)$) obtained in [11], we can formulate the following result.

PROPOSITION 2. *In Variant (C) for $C(T)$ the following inequalities are valid:*

$$F(T^{-1}) - \Delta(T^{-1}) \leq C(T) \leq F(T^{-1}).$$

At the same time for small T

$$\frac{T}{2} + O(T^2) \leq C(T) \leq T + O(T^2)$$

and for large T

$$(13) \quad \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T) \leq C(T) \leq \log T - \mathbb{C} + O(T^{-1} \log^2 T).$$

From (13) we obtain that for large T (actually this case is the main interest for quickest detection problems) the minimax risk $C(T)$ behaves as $\log T$ (in the sense that $C(T)/\log T \rightarrow 1$, $T \rightarrow \infty$; asymptotics of the first order) at the same time that the difference between the second terms (i.e., $-(1 + \mathbb{C})$ and $-\mathbb{C}$) is equal to 1. The next terms tend to 0 as $O(T^{-1} \log^2 T)$, $T \rightarrow \infty$.

1.6. The main goal of the present paper is to eliminate the “gap” between the second terms in the left- and right-hand sides of (13).

For this goal we consider the smaller quantity $\overline{C}(T)$, which is the minimax risk (see (3)) over the class $\overline{\mathfrak{M}}_T$ of randomized stopping times, instead of $C(T)$. (In the case of the quickest detection problems in discrete time, randomized stopping times were considered in [20].) It follows from the results obtained in what follows that for large T the following inequalities hold:

$$(14) \quad \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T) \leq \overline{C}(T) \leq \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T).$$

From (14) we obtain that for $\overline{C}(T)$ the *asymptotics of the second order*

$$\overline{C}(T) = \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T), \quad T \rightarrow \infty,$$

holds. Of course from this formula more rough asymptotics follows:

$$\frac{\overline{C}(T)}{\log T - (1 + \mathbb{C})} \longrightarrow 1, \quad T \rightarrow \infty.$$

2. On the method for obtaining lower and upper estimates for $\overline{C}(T)$.

2.1. Similarly to (12) we obtain the inequality $\overline{B}(T) \leq \overline{C}(T)$.

For $B(T)$ representation (4) is valid, and we obtain a similar representation (using the same method as in [11]) for $\overline{B}(T)$:

$$\overline{B}(T) = \inf_{\tau \in \overline{\mathfrak{M}}_T} \frac{1}{T} \overline{\mathbf{E}}_\infty \int_0^\tau \psi_t(\overline{\omega}) dt \quad \text{for all } T > 0,$$

where $\psi_t(\overline{\omega}) = \psi_t(\omega)$ for $\overline{\omega} = (\omega, \tilde{\omega})$.

The process $\psi = (\psi_t(\omega))_{t \geq 0}$ with respect to the measure $\mathbf{P}_\infty$ is a diffusion Markov process with the differential

$$d\psi_t(\omega) = dt + \frac{\mu}{\sigma} \psi_t(\omega) dB_t(\omega), \quad \psi_0(\omega) = 0.$$

From the general optimal stopping theory for Markov processes, we obtain

$$(15) \quad \overline{B}(T) = B(T).$$

In other words if we take the infimum in (6) over the wider class $\overline{\mathfrak{M}}_T$ of randomized stopping times, we obtain the same result as if we took the infimum over the class $\mathfrak{M}_T$. (In Markov optimal stopping problems $\inf_{\tau \in \mathfrak{M}_T}$ coincides with $\inf_{\tau \in \overline{\mathfrak{M}}_T}$; see [8, Chap. II, section 7, p. 100].) Using the inequality $\overline{B}(T) \leq \overline{C}(T)$ and (7), from (15) we obtain the lower estimate for $\overline{C}(T)$: for all $T > 0$

$$(16) \quad F(T^{-1}) - \Delta(T^{-1}) \leq \overline{C}(T).$$

2.2. In order to get the upper estimate for $\overline{C}(T)$ we use the following considerations.

It is clear that, for a particular stopping time τ^* from $\overline{\mathfrak{M}}_T$, the inequality

$$(17) \quad \overline{C}(T) \leq \sup_{\theta \geq 0} \overline{\mathbf{E}}_\theta(\tau^* - \theta \mid \tau^* \geq \theta)$$

holds.

Thus in order to decrease the “gap” between the upper and lower estimates we should find the stopping time $\tau^* \in \overline{\mathfrak{M}}_T$ such that the value of $\sup_\theta \overline{\mathbf{E}}_\theta(\tau^* - \theta \mid \tau^* \geq \theta)$ would be as close to the value $\overline{B}(T)$ ($= B(T)$) as possible. As we will show, the stopping time $\tau_{g_A}^*$ from $\overline{\mathfrak{M}}_T$ with the specially chosen value of $A = A(T)$ “almost” satisfies the desired property to decrease the “gap” at least for large values of T .

The stopping time $\tau_{g_A}^*$ is constructed in the following way. Let $\psi^* = (\psi_t^*(\overline{\omega}))_{t \geq 0}$ be a process with the stochastic differential

$$(18) \quad d\psi_t^*(\overline{\omega}) = dt + \frac{\mu}{\sigma^2} \psi_t^*(\overline{\omega}) dX_t(\overline{\omega});$$

let $\mathbb{L}_\infty$ be an adjoint infinitesimal operator (see [4]) of the process $\psi = (\psi_t)_{t \geq 0}$ with the differential $d\psi_t = dt + (\mu/\sigma) \psi_t dB_t$, $\psi_0 = 0$ ($\rho \equiv \mu^2/(2\sigma^2) = 1$). Obviously, $\mathbb{L}_\infty = \partial^2(y^2 \cdot)/\partial y^2 - \partial/\partial y$. Denote by $g_A(y)$, $0 \leq y \leq A$, a probability density function satisfying, for $0 \leq y < A$, the equation

$$(19) \quad \mathbb{L}_\infty g_A = 0,$$

boundary condition $g_A(A) = 0$, and the normalizing condition $\int_0^A g_A(y) dy = 1$.

Let the initial value $\psi_0^*(\bar{\omega})$ of the process $\psi^* = (\psi_t^*(\bar{\omega}))_{t \geq 0}$ be random and independent of the Brownian motion $B(\bar{\omega}) = (B_t(\omega, \bar{\omega}))_{t \geq 0}$ ($B_t(\omega, \bar{\omega}) = B_t(\omega)$, $t \geq 0$), and let it be distributed according to the distribution $G_A(y) = \bar{\mathbf{P}}\{\psi_0^*(\bar{\omega}) \leq y\}$, $y \in [0, A]$, where $G_A(y) = \int_0^y g_A(x) dx$. Set

$$(20) \quad \bar{\tau}_{g_A}^* = \inf\{t \geq 0: \psi_t^*(\bar{\omega}) \geq A\}.$$

Using (17) and the *independence* of the initial value $\psi_0^*(\bar{\omega})$ of the Brownian motion $B = (B_t(\bar{\omega}))_{t \geq 0}$, we get that for the stopping time $\bar{\tau}_{g_A}^*$ the inequalities

$$\begin{aligned} \bar{C}(T) &\leq \sup_{\theta} \bar{\mathbf{E}}_{\theta}(\bar{\tau}_{g_A}^* - \theta \mid \bar{\tau}_{g_A}^* \geq \theta) \\ &= \sup_{\theta} \int_0^A \mathbf{E}_{\theta}(\tau_A^{(y)}(\omega) - \theta \mid \tau_A^{(y)}(\omega) \geq \theta) g_A(y) dy \\ &\leq \int_0^A \sup_{\theta} \{\mathbf{E}_{\theta}(\tau_A^{(y)}(\omega) - \theta \mid \tau_A^{(y)}(\omega) \geq \theta)\} g_A(y) dy \\ (21) \quad &= \int_0^A \mathbf{E}_0 \tau_A^{(y)}(\omega) g_A(y) dy = \bar{\mathbf{E}}_0 \bar{\tau}_{g_A}^* \end{aligned}$$

are valid, where

$$\tau_A^{(y)}(\omega) = \inf\{t \geq 0: \psi_t^{(y)}(\omega) \geq A\}, \quad d\psi_t^{(y)}(\omega) = dt + \frac{\mu}{\sigma^2} \psi_t^{(y)}(\omega) dX_t(\omega)$$

with $\psi_0^{(y)} = y$, $0 \leq y \leq A$ (the first equality in (21) follows from the proof of Theorem 3.1 of [11]).

Thus we obtain the following upper and lower estimates for $\bar{C}(T)$ for all $T > 0$:

$$(22) \quad F(T^{-1}) - \Delta(T^{-1}) \leq \bar{C}(T) \leq \bar{\mathbf{E}}_0 \bar{\tau}_{g_A}^*,$$

where the threshold $A = A(T)$ is chosen such that $\bar{\tau}_{g_A}^* \in \bar{\mathfrak{M}}_T$, i.e.,

$$\bar{\mathbf{E}}_{\infty} \bar{\tau}_{g_A}^* = T.$$

Therefore,

$$(23) \quad \bar{\mathbf{E}}_0 \bar{\tau}_{g_A}^* = \int_0^A \mathbf{E}_0 \tau_A^{(y)}(\omega) g_A(y) dy, \quad \bar{\mathbf{E}}_{\infty} \bar{\tau}_{g_A}^* = \int_0^A \mathbf{E}_{\infty} \tau_A^{(y)}(\omega) g_A(y) dy.$$

Thus in order to find the upper bound $\bar{\mathbf{E}}_0 \bar{\tau}_{g_A}^*$ in (22) first we should find the randomizing distribution $G_A = G_A(y)$, $0 \leq y \leq A$, with the density $g_A(y)$ and then find $\bar{\mathbf{E}}_{\infty} \bar{\tau}_{g_A}^*$. This will be done in sections 3 and 4.

3. Randomizing density g_A .

3.1. The density $g_A = g_A(y)$, $0 \leq y \leq A$, is defined as a positive solution of (19) with the boundary condition $g_A(A) = 0$ and the normalizing condition $\int_0^A g_A(y) dy = 1$. In the following lemma we will find an explicit formula for this density.

LEMMA 1. *The density $g_A = g_A(y)$, $0 \leq y \leq A$, is defined by the relation*

$$(24) \quad g_A(y) = f(y) \frac{h(A) - h(y)}{A},$$

where

$$(25) \quad f(y) = y^{-2} e^{1/y}, \quad h(y) = ye^{1/y} - \text{Ei}(y^{-1}),$$

and the integral exponential function $-\text{Ei}(z)$ is defined as follows (see [2]):

$$(26) \quad -\text{Ei}(z) = \begin{cases} \int_{-z}^{\infty} \frac{e^{-t}}{t} dt, & z < 0, \\ \lim_{\varepsilon \rightarrow +0} \left[\int_{-z}^{-\varepsilon} \frac{e^{-t}}{t} dt + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt \right], & z > 0. \end{cases}$$

(If $z > 0$, then $-\text{Ei}(-z) = \int_z^{\infty} t^{-1} e^{-t} dt$, which we have already mentioned in Proposition 1.)

Proof. Fix $A > 0$ and write g instead of g_A . Integrating (19), we obtain that $g = (y^2 g)' + C_1$. Hence,

$$(27) \quad g' = g(y^{-2} - 2y^{-1}) - C_1 y^{-2}.$$

Let us represent a solution of this equation in the form

$$(28) \quad g(y) = f(y) u(y),$$

where $f(y) = y^{-2} e^{-1/y}$. It follows that

$$(29) \quad g'(y) = (f(y) u(y))' = f(y) u(y) (y^{-2} - 2y^{-1}) + f(y) u'(y).$$

From (27) and (28) we obtain

$$(30) \quad g'(y) = f(y) u(y) (y^{-2} - 2y^{-1}) - C_1 y^{-2}.$$

Comparing this expression with (29), we obtain that $f(y) u'(y) = -C_1/y^2$. Therefore $u'(y) = -C_1 e^{1/y}$ and

$$\begin{aligned} u(y) &= C_2 - C_1 \int_0^y e^{1/z} dz = C_2 + C_1 \int_{\infty}^{1/y} \frac{e^t}{t^2} dt = C_2 - C_1 \int_{1/y}^{\infty} \frac{e^t}{t^2} dt \\ &= C_2 + C_1 [-ye^{1/y} + \text{Ei}(y^{-1})]. \end{aligned}$$

Thus $g(y) = f(y) u(y)$ and we obtain that

$$(31) \quad g(y) = C_1 [-y^{-1} + f(y) \text{Ei}(y^{-1})] + C_2 f(y).$$

Let us determine the constants C_1 and C_2 using the conditions $g(A) = 0$ and $\int_0^A g(y) dy = 1$.

To this aim we denote

$$I_1 = \int_0^A [-y^{-1} + f(y) \text{Ei}(y^{-1})] dy, \quad I_2 = \int_0^A f(y) dy.$$

Hence the condition

$$\int_0^A g(y) dy = 1$$

takes the form

$$C_1 I_1 + C_2 I_2 = 1.$$

Let us find I_1 and I_2 .

We have

$$(32) \quad I_1 = \int_0^A [-y^{-1} + y^{-2} \operatorname{Ei}(y^{-1}) e^{-1/y}] dy.$$

It is well known (see [2, formula 8.212.2]) that $\operatorname{Ei}(z) = e^z(z^{-1} + \int_0^\infty (z-t)^{-2} e^{-t} dt)$ for $z > 0$. Therefore,

$$y^{-2} \operatorname{Ei}(y^{-1}) e^{-1/y} = y^{-1} + y^{-2} \int_0^\infty \frac{e^{-t} dt}{(1/y - t)^2},$$

and from (32) we obtain

$$\begin{aligned} I_1 &= \int_0^A \frac{1}{y^2} \left[\int_0^\infty \frac{e^{-t}}{(1/y - t)^2} dt \right] dy = \int_0^\infty e^{-t} \left[\int_0^A \frac{dy}{(1 - yt)^2} \right] dt \\ &= \int_0^\infty \frac{e^{-t}}{t^2} \left[\int_0^A \frac{dy}{(1/t - y)^2} \right] dt = \int_0^\infty \frac{e^{-t}}{t^2} \left(\frac{1}{1/t - A} - \frac{1}{1/t} \right) dt \\ &= \int_0^\infty \frac{e^{-t}}{1/A - t} dt. \end{aligned}$$

Using the well-known formula

$$(33) \quad \operatorname{Ei}(\pm xy) = \pm e^{\pm xy} \int_0^\infty \frac{e^{-xt}}{y \mp t} dt, \quad \operatorname{Re} y > 0, \quad x > 0$$

(see [2, formula 8.212.5]), we have

$$I_1 = e^{-1/A} \operatorname{Ei}(A^{-1}).$$

For I_2 we obtain

$$I_2 = \int_0^A f(y) dy = \int_0^A y^{-2} e^{-1/y} dy = e^{-1/A}.$$

Thus the condition $C_1 I_1 + C_2 I_2 = 1$ (i.e., the condition $\int_0^A g(y) dy = 1$) takes the form $C_1 e^{-1/A} \operatorname{Ei}(A^{-1}) + C_2 e^{-1/A} = 1$, and the condition $g(A) = 0$ takes the form $C_1(-Ae^{1/A} + \operatorname{Ei}(A^{-1})) + C_2 = 0$.

Using these two equalities, we obtain that

$$C_1 = A^{-1}, \quad C_2 = e^{1/A} - \operatorname{Ei}(A^{-1}) A^{-1},$$

and (31) can be transformed into (24). This completes the proof of Lemma 1.

3.2. Denote by $Q_A(y)$ the probability measure on the set $[0, A]$ such that for the process $\psi^* = (\psi_t^*(\bar{\omega}))_{t \geq 0}$ with the stochastic differential (18) and the random initial value $\psi_0^*(\bar{\omega})$, which is independent of the Brownian motion $B(\bar{\omega}) = (B_t(\omega, \tilde{\omega}))_{t \geq 0}$ (where $B_t(\omega, \tilde{\omega}) = B_t(\omega)$, $t \geq 0$) and distributed according to the measure $Q_A(y)$ (i.e., $\bar{\mathbf{P}}\{\psi_0^*(\bar{\omega}) \leq y\} = Q_A(y)$, $y \in [0, A]$), for $t > 0$, the following equality holds:

$$(34) \quad \bar{\mathbf{P}}_\infty(\psi_t^*(\bar{\omega}) \leq y \mid \psi_s^*(\bar{\omega}) < A, s \leq t) = Q_A(y).$$

The distribution $Q_A(y)$, $y \in [0, A]$, of the initial value $\psi_0^*(\bar{\omega})$ with property (34) is called *quasi-stationary* (see, for example, [9], [10], [13], [14], [15], [16], [17], [18], [23]).

Using the results of [9] (see also [10], [14], [17], [23]), we obtain that the quasi-stationary distribution $Q_A(y)$ has the density $q_A(y)$, which is a solution of the equation

$$(35) \quad \mathbb{L}_\infty q_A(y) = -\lambda_1 q_A(y), \quad y \in [0, A],$$

where $\mathbb{L}_\infty = \partial^2(y^2 \cdot) / \partial y^2 - \partial / \partial y$ is the adjoint infinitesimal operator of the process $\psi = (\psi_t)_{t \geq 0}$ with the differential $d\psi_t = dt + (\mu/\sigma) \psi_t dB_t$, $\psi_0 = 0$ ($\rho \equiv \mu^2 / (2\sigma^2) = 1$), and $\lambda_1 = \lambda_1(A) > 0$ is the smallest eigenvalue of the operator $\mathbb{L}_\infty$ (in the case under consideration this operator will have a purely discrete spectrum). Moreover, the density $q_A(y)$ should satisfy the boundary conditions $q_A(A) = 0$, $q'_A(A) = -A^{-2}e^{1-1/A}$ and the normalizing condition $\int_0^A q_A(y) dy = 1$.

Using the results of [9] and [18], we obtain

$$(36) \quad \lim_{A \rightarrow \infty} Q_A(y) = Q_\infty(y), \quad y > 0,$$

where $Q_\infty(y) = \lim_{t \rightarrow \infty} \mathbf{P}_\infty\{\psi_t \leq y\}$, $y > 0$, is a *stationary (invariant)* distribution of the process $\psi = (\psi_t)_{t \geq 0}$ (the definition of the process is given above). The stationary distribution $Q_\infty(y)$ also has the density $q_\infty(y)$, which is a solution of the equation

$$(37) \quad \mathbb{L}_\infty q_\infty(y) = 0, \quad y > 0,$$

with the boundary condition $\lim_{y \rightarrow \infty} \{(y^2 q_\infty(y))' - q_\infty(y)\} = 0$ and the normalizing condition $\int_0^\infty q_\infty(y) dy = 1$ (see also [11], [19]). Thus using (35)–(37), we obtain that

$$\lim_{A \rightarrow \infty} \lambda_1(A) = 0.$$

Therefore, a solution $q_A(y)$ of (35) can be expanded in powers of a small parameter $\lambda_1(A)$ as $A \rightarrow \infty$; i.e., we represent $q_A(y)$ as a series $q_A(y) = q_A^0(y) + \lambda_1(A) q_A^1(y) + \dots$, $y \in [0, A]$, where the functions $q_A^0(y)$ and $q_A^1(y)$ are solutions of the equations

$$\mathbb{L}_\infty q_A^0(y) = 0, \quad \mathbb{L}_\infty q_A^1(y) = -q_A^0(y), \quad y \in [0, A],$$

with boundary conditions $q_A^0(A) = 0$ and $q_A^1(A) = 0$ and normalizing conditions $\int_0^A q_A^0(y) dy = 1$ and $\int_0^A q_A^1(y) dy = 0$, respectively. Thus the density $g_A(y) (\equiv q_A^0(y))$ of the randomizing distribution $G_A(y)$ is the zero term in the expansion for the quasi-stationary distribution in powers of the first eigenvalue $\lambda_1(A) \rightarrow 0$ of the adjoint infinitesimal operator $\mathbb{L}_\infty$ of the process $\psi = (\psi_t)_{t \geq 0}$ as $A \rightarrow \infty$.

Using the expansion $q_A(y)$ in powers of $\lambda_1(A)$ as $A \rightarrow \infty$ and the condition $q'_A(A) = -A^{-2}e^{1-1/A}$, we obtain an asymptotics

$$\lambda_1(A) = 6eA^{-1} + O(A^{-2}), \quad A \rightarrow \infty.$$

Remark 1. If for the stopping time $\bar{\tau}_{g_A}^*$ from (20) the initial value of the process $\psi^* = (\psi_t^*(\bar{\omega}))_{t \geq 0}$ is randomized without using the distribution $G_A(y)$ only using the distribution $Q_A(y)$, then from (34) we obtain the equality (cf. with (21))

$$\sup_{\theta} \bar{\mathbf{E}}_\theta(\bar{\tau}_{g_A}^* - \theta \mid \bar{\tau}_{g_A}^* \geq \theta) = \bar{\mathbf{E}}_0 \bar{\tau}_{g_A}^*, \quad \theta \geq 0.$$

Remark 2. It is easy to calculate that

$$q_\infty(y) = y^{-2} e^{-1/y}, \quad y > 0;$$

i.e., for the limiting distribution $Q_\infty(y) = \int_0^y q_\infty(x) dx$ we obtain that

$$Q_\infty(y) = e^{-1/y}, \quad y > 0.$$

In the theory of extreme value distributions this is the well-known “Fréchet-type distribution” (see [12]).

Obviously, $C_1 = C_1(A)$ and $C_2 = C_2(A)$ are such that $C_1(A) \rightarrow 0$, $C_2(A) \rightarrow 1$ as $A \rightarrow \infty$. Thus using (24), we obtain that

$$\lim_{A \rightarrow \infty} g_A(y) = q_\infty(y),$$

as it should be.

4. Calculation of $\bar{\mathbf{E}}_\infty \bar{\tau}_{g_A}^*$.

4.1. According to the second equality in (23), in order to calculate $\bar{\mathbf{E}} \bar{\tau}_{g_A}^*$ we have to find $\mathbf{E}_\infty \tau_A^{(y)}$, where for $0 \leq y \leq A$

$$\tau_A^{(y)} = \inf\{t \geq 0: \psi_t^{(y)} \geq A\}$$

with the process $(\psi_t^{(y)})_{t \geq 0}$, for which

$$(38) \quad \psi_t^{(y)} = y + t + \frac{\mu}{\sigma} \int_0^t \psi_s^{(y)} dB_s, \quad t \geq 0.$$

Since $\psi_{t \wedge \tau_A^{(y)}}^{(y)} \leq A$ and the process $(\int_0^t \psi_s^{(y)} dB_s)_{t \geq 0}$ is a local martingale, it follows that $\mathbf{E}_\infty \int_0^{t \wedge \tau_A^{(y)}} \psi_s^{(y)} dB_s = 0$. Using (38), we obtain that $y + \mathbf{E}_\infty(t \wedge \tau_A^{(y)}) \leq A$ for any $t > 0$. Thus $\mathbf{E}_\infty(t \wedge \tau_A^{(y)}) \leq A - y$. Passing to the limit as $t \rightarrow \infty$ in this inequality, we obtain that (by the monotone convergence theorem)

$$(39) \quad \mathbf{E}_\infty \tau_A^{(y)} \leq A - y < \infty.$$

Using (38) and (39) and applying “the first Wald’s identity” (see [5]) to the martingale $(\int_0^{t \wedge \tau_A^{(y)}} \psi_s^{(y)} dB_s)_{t \geq 0}$, we obtain that

$$(40) \quad \mathbf{E}_\infty \psi_{\tau_A^{(y)}}^{(y)} = y + \mathbf{E}_\infty \tau_A^{(y)}.$$

Using the comparison theorem for stochastic differential equations (see [1]), we have that $\psi_t^{(y)} \geq L_t^{(y)}$ ($\mathbf{P}$ -a.s.), $t \geq 0$, where $L_t^{(y)}$ has the differential

$$(41) \quad dL_t^{(y)} = \frac{\mu}{\sigma} L_t^{(y)} dB_t, \quad L_0^{(y)} = y.$$

Since the solution of (41) has the form

$$L_t^{(y)} = y \exp \left\{ \frac{\mu}{\sigma} B_t - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right\},$$

it follows that for $y > 0$ the process $L_t^{(y)}$ with probability 1 reaches any level $A > 0$. Thus the process $\psi_t^{(y)}$, $t \geq 0$, also reaches any level $A > 0$ with probability 1. Therefore, $\psi_{\tau_A^{(y)}}^{(y)} = A$ ($\mathbf{P}$ -a.s.) and from (40) we obtain that

$$\mathbf{E}_\infty \tau_A^{(y)} = A - y$$

for any $y > 0$. (By continuity this formula also holds for $y = 0$.)

Thus,

$$\overline{\mathbf{E}}_\infty \overline{\tau}_{g_A}^* = \int_0^A \mathbf{E}_\infty \tau_A^{(y)} dG_A(y) = A - \int_0^A y g_A(y) dy.$$

4.2. Using (24), (25), we obtain that

$$g_A(y) = \frac{1}{y^2} e^{-1/y} \frac{(Ae^{1/A} - \text{Ei}(1/A)) - (ye^{1/y} - \text{Ei}(1/y))}{A}.$$

Thus,

$$\begin{aligned} \int_0^A y g_A(y) dy &= \left(e^{-1/A} - \frac{1}{A} \text{Ei}\left(\frac{1}{A}\right) \right) \int_0^A \frac{1}{y} e^{-1/y} dy \\ &\quad + \frac{1}{A} \int_0^A \left(-1 + \frac{1}{y} \text{Ei}\left(\frac{1}{y}\right) \right) e^{-1/y} dy. \end{aligned} \quad (42)$$

We have

$$J_1 \equiv \int_0^A \frac{1}{y} e^{-1/y} dy = \int_{1/A}^\infty \frac{e^{-z}}{z} dz = -\text{Ei}\left(-\frac{1}{A}\right). \quad (43)$$

Let us show that

$$J_2 \equiv \int_0^A \frac{1}{y} \text{Ei}\left(\frac{1}{y}\right) e^{-1/y} dy = -\text{Ei}\left(-\frac{1}{A}\right) \text{Ei}\left(\frac{1}{A}\right) + \int_0^\infty e^{-z/A} \frac{\log(1+z)}{z} dz. \quad (44)$$

Actually,

$$\begin{aligned} J_2 &= \int_0^A \frac{1}{y} \text{Ei}\left(\frac{1}{y}\right) e^{-1/y} dy = \int_{1/A}^\infty \frac{e^{-z}}{z} \text{Ei}(z) dz \\ &= \int_{1/A}^\infty \text{Ei}(z) d(\text{Ei}(-z)) = \text{Ei}(-z) \text{Ei}(z) \Big|_{1/A}^\infty - \int_{1/A}^\infty \text{Ei}(-z) \frac{e^z}{z} dz. \end{aligned}$$

By L'Hospital's theorem and the formulas $\lim_{z \rightarrow \infty} \text{Ei}(-z) = 0$, $\lim_{z \rightarrow \infty} \text{Ei}(z) = \infty$, $(\text{Ei}(-z))' = e^{-z}/z$, and $(\text{Ei}(z))' = e^z/z$, $z > 0$, we obtain that

$$\begin{aligned} \lim_{z \rightarrow \infty} \text{Ei}(-z) \text{Ei}(z) &= \lim_{z \rightarrow \infty} \frac{\text{Ei}(z)}{(\text{Ei}(-z))^{-1}} = \lim_{z \rightarrow \infty} \frac{-e^z/z}{(e^{-z}/z)(\text{Ei}(-z))^{-2}} \\ &= - \lim_{z \rightarrow \infty} e^{2z} (\text{Ei}(-z))^2 = - \lim_{z \rightarrow \infty} (-z^{-1} + O(z^{-2}))^2 = 0, \end{aligned} \quad (45)$$

where we use relation 3.2.4 of [3], according to which

$$\lim_{z \rightarrow \infty} e^z \operatorname{Ei}(-z) = \lim_{z \rightarrow \infty} (-z^{-1} + O(z^{-2})) = 0.$$

Thus,

$$J_2 = -\operatorname{Ei}\left(-\frac{1}{A}\right) \operatorname{Ei}\left(\frac{1}{A}\right) + \int_{1/A}^{\infty} (-\operatorname{Ei}(-z)) \frac{e^z}{z} dz.$$

By (33) we have

$$\int_{1/A}^{\infty} (-\operatorname{Ei}(-z)) \frac{e^z}{z} dz = \int_{1/A}^{\infty} e^{-z} \left(\int_0^{\infty} \frac{e^{-t}}{z+t} dt \right) \frac{e^z}{z} dz = \int_0^{\infty} e^{-t} K(t) dt,$$

where $K(t) = \int_{1/A}^{\infty} [z(z+t)]^{-1} dz = t^{-1} \log(1+At)$.

Therefore,

$$\begin{aligned} J_2 &= -\operatorname{Ei}\left(-\frac{1}{A}\right) \operatorname{Ei}\left(\frac{1}{A}\right) + \int_0^{\infty} \frac{e^{-t}}{t} \log(1+At) dt \\ &= -\operatorname{Ei}\left(-\frac{1}{A}\right) \operatorname{Ei}\left(\frac{1}{A}\right) + \int_0^{\infty} e^{-z/A} \frac{\log(1+z)}{z} dz. \end{aligned}$$

Thus we obtain (44).

By (42)–(44), we obtain that

$$\begin{aligned} \int_0^A yg_A(y) dy &= (e^{1/A} - A^{-1} \operatorname{Ei}(A^{-1})) J_1 - 1 + A^{-1} J_2 \\ &= -1 - (e^{1/A} - A^{-1} \operatorname{Ei}(A^{-1})) \operatorname{Ei}(-A^{-1}) \\ &\quad + A^{-1} \left(-\operatorname{Ei}(-A^{-1}) \operatorname{Ei}(A^{-1}) + \int_0^{\infty} e^{-z/A} \frac{\log(1+z)}{z} dz \right) \\ &= e^{1/A} (-\operatorname{Ei}(-A^{-1})) - \left(1 - A^{-1} \int_0^{\infty} e^{-z/A} \frac{\log(1+z)}{z} dz \right) \\ &= F(A^{-1}) - \Delta(A^{-1}), \end{aligned}$$

where $F(b)$ and $\Delta(b)$ are defined as in (8). In [11] it was shown that

$$(46) \quad F(A^{-1}) = e^{1/A} (-\operatorname{Ei}(-A^{-1})) = \begin{cases} A + O(A^2), & A \rightarrow 0, \\ \log A - \mathbb{C} + O(A^{-1} \log A), & A \rightarrow \infty, \end{cases}$$

where $\mathbb{C} = 0.577 \dots$ is Euler's constant and

(47)

$$\Delta(A^{-1}) = 1 - A^{-1} \int_0^{\infty} e^{-z/A} \frac{\log(1+z)}{z} dz = \begin{cases} \frac{A}{2} + O(A^2), & A \rightarrow 0, \\ 1 + O(A^{-1} \log^2 A), & A \rightarrow \infty. \end{cases}$$

Thus we have proved the following lemma.

LEMMA 2. For any $A > 0$,

$$(48) \quad \overline{\mathbf{E}}_{\infty} \overline{\tau}_{g_A}^* = A - [F(A^{-1}) - \Delta(A^{-1})].$$

For small and large values of A ,

$$\overline{\mathbf{E}}_{\infty} \overline{\tau}_{g_A}^* = \begin{cases} \frac{A}{2} + O(A^2), & A \rightarrow 0, \\ A - \log A + (1 + \mathbb{C}) + O(A^{-1} \log^2 A), & A \rightarrow \infty. \end{cases}$$

4.3. Since the stopping time $\overline{\tau}_{g_A}^*$ should belong to the class $\overline{\mathfrak{M}}_T$ for some $T > 0$, it follows that the threshold $A = A(T)$ should be chosen such that $\overline{\mathbf{E}}_{\infty} \overline{\tau}_{g_{A(T)}}^* = T$; i.e., from (48) we obtain that the threshold $A = A(T)$ is defined as a solution of the equation

$$A - [F(A^{-1}) - \Delta(A^{-1})] = T.$$

From (46) and (47) by iterations we obtain that for small and large T the following asymptotics for $A(T)$ holds:

$$A(T) = \begin{cases} 2T + O(T^2), & T \rightarrow 0, \\ T + [\log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T)], & T \rightarrow \infty. \end{cases}$$

5. Calculation of $\overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^*$ and estimates for $\overline{\mathcal{C}}(T)$.

5.1. The following lemma holds.

LEMMA 3. For any $A > 0$,

$$(49) \quad \overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^* = F(A^{-1}) - \Delta(A^{-1}).$$

For small and large A ,

$$\overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^* = \begin{cases} \frac{A}{2} + O(A^2), & A \rightarrow 0, \\ \log A - (1 + \mathbb{C}) + O(A^{-1} \log^2 A), & A \rightarrow \infty. \end{cases}$$

Proof. Using the first equality in (23), we obtain that $\overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^* = \int_0^A \mathbf{E}_0 \tau_A^{(y)} g_A(y) dy$. In [11] it was proved that $\mathbf{E}_0 \tau_A^{(y)} = F(A^{-1}) - F(y^{-1})$, where $F(b) = e^b(-\text{Ei}(-b))$. Therefore,

$$\overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^* = F(A^{-1}) - \int_0^A F(y^{-1}) g_A(y) dy.$$

By Lemma 1, it follows that

$$g_A(y) = -\frac{1}{Ay} + \frac{e^{-1/y} \text{Ei}(y^{-1})}{Ay^2} + (e^{1/A} - A^{-1} \text{Ei}(A^{-1})) y^{-2} e^{-1/y}.$$

Using the equality $F(y^{-1}) = e^{1/y}(-\text{Ei}(-y^{-1}))$, we obtain that

$$(50) \quad \int_0^A F(y^{-1}) g_A(y) dy = -A^{-1} K_1 + A^{-1} K_2 + (e^{1/A} - A^{-1} \text{Ei}(A^{-1})) K_3,$$

where

$$K_1 = \int_0^A \frac{1}{y} e^{1/y} (-\text{Ei}(-y^{-1})) dy, \quad K_2 = \int_0^A y^{-2} \text{Ei}(y^{-1}) (-\text{Ei}(-y^{-1})) dy, \\ K_3 = \int_0^A y^{-2} (-\text{Ei}(-y^{-1})) dy.$$

Similar to the calculations for J_2 (see (44)), we obtain that

$$K_1 = - \int_{1/A}^{\infty} \frac{e^z}{z} \operatorname{Ei}(-z) dz = \int_0^{\infty} e^{-z/A} \frac{\log(1+z)}{z} dz.$$

Further,

$$\begin{aligned} K_2 &= - \int_{1/A}^{\infty} \operatorname{Ei}(z) \operatorname{Ei}(-z) dz = - \int_{1/A}^{\infty} \operatorname{Ei}(z) d(\operatorname{Ei}(-z) z + e^{-z}) \\ &= - \operatorname{Ei}(z) (z \operatorname{Ei}(-z) + e^{-z}) \Big|_{1/A}^{\infty} + \int_{1/A}^{\infty} \left(e^z \operatorname{Ei}(-z) + \frac{1}{z} \right) dz. \end{aligned}$$

Analogously to (45), we have that $\lim_{z \rightarrow \infty} \operatorname{Ei}(z) (z \operatorname{Ei}(-z) + e^{-z}) = 0$. Using the representation $e^z \operatorname{Ei}(-z) + z^{-1} = \int_0^{\infty} (z+t)^{-2} e^{-t} dt$ (see [2, formula 8.212.3]), we obtain that

$$\int_{1/A}^{\infty} (e^z \operatorname{Ei}(-z) + z^{-1}) dz = - \operatorname{Ei}(-A^{-1}) e^{1/A}.$$

Thus,

$$K_2 = A^{-1} \operatorname{Ei}(A^{-1}) \operatorname{Ei}(-A^{-1}) + e^{-1/A} \operatorname{Ei}(A^{-1}) - e^{1/A} \operatorname{Ei}(-A^{-1}).$$

Finally,

$$K_3 = \int_0^A y^{-2} (-\operatorname{Ei}(-y^{-1})) dy = - \int_{1/A}^{\infty} \operatorname{Ei}(-z) dz = A^{-1} \operatorname{Ei}(-A^{-1}) + e^{-1/A}.$$

From (50) and the representations for K_1 , K_2 , and K_3 , after simple calculations we obtain that $\int_0^A F(y^{-1}) g_A(y) dy = \Delta(A^{-1})$, and therefore,

$$\overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^* = F(A^{-1}) - \int_0^A F(A^{-1}) g_A(y) dy = F(A^{-1}) - \Delta(A^{-1}).$$

Lemma 3 is proved.

5.2. Let $A = A(T)$ be chosen in such a way that the equality $\overline{\mathbf{E}}_{\infty} \overline{\tau}_{g_A(T)}^* = T$ holds for $T > 0$.

Using (16), the obvious estimation $\overline{\mathcal{C}}(T) \leq \overline{\mathbf{E}}_0 \overline{\tau}_{g_A}^*$, and Lemmas 2 and 3, we obtain the following main result of the present paper.

THEOREM. *For any $T > 0$, the minimax risk*

$$\overline{\mathcal{C}}(T) = \inf_{\overline{\tau} \in \overline{\mathfrak{M}}_T} \sup_{\theta \geq 0} \mathbf{E}_{\theta}(\overline{\tau} - \theta \mid \overline{\tau} \geq \theta)$$

can be estimated from above and below in the following way:

$$(51) \quad F(T^{-1}) - \Delta(T^{-1}) \leq \overline{\mathcal{C}}(T) \leq F(A^{-1}) - \Delta(A^{-1}) \quad (= \overline{\mathcal{C}}^*(T)),$$

where $A = A(T)$ is a root of the equation

$$A - [F(A^{-1}) - \Delta(A^{-1})] = T.$$

FIG. 1. Plots of the estimates from above $C^*(T)$ and from below $B(T)$ of the minimax risk $C(T)$.

FIG. 2. Plots of the estimates from above $\bar{C}^*(T)$ and from below $B(T)$ of the minimax risk $\bar{C}(T)$.

For a small T ,

$$(52) \quad \frac{T}{2} + O(T^2) \leq \bar{C}(T) \leq T + O(T^2).$$

For large T ,

$$(53) \quad \bar{C}(T) = \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T),$$

where $\mathbb{C} = 0.577 \dots$ is Euler's constant.

Proof. Since $B(T) \leq \bar{C}(T)$ and $B(T) = T/2 + O(T^2)$ (see (9)), we obtain the left-hand side inequality in (52). For a small A the function $F(A^{-1}) - \Delta(A^{-1}) = B(A)$ behaves like $A/2 + O(A^2)$. Thus from (51), (47) for $A = A(T)$, and a small T , we obtain that

$$\bar{C}(T) \leq A(T) - \frac{A(T)}{2} + O(A^2(T)) = 2T - \frac{2T}{2} + O(T^2) = T + O(T^2),$$

which proves the right-hand side inequality in (52).

In order to prove (53) we will use that $B(T) \leq \overline{C}(T) \leq \overline{\mathbf{E}}_0 \overline{\tau}_{g_{A(T)}}^*$ as it was proved earlier. For large T from (9) we obtain that

$$(54) \quad B(T) = \log T - 1 - \mathbb{C} + O(T^{-1} \log^2 T).$$

Using (49), we obtain

$$\overline{\mathbf{E}}_0 \overline{\tau}_{g_{A(T)}}^* = F\left(\frac{1}{A(T)}\right) - \Delta\left(\frac{1}{A(T)}\right) = B(A(T)).$$

As $T \rightarrow \infty$,

$$A(T) = T + [\log T - 1 - \mathbb{C} + O(T^{-1} \log^2 T)],$$

and, therefore,

$$(55) \quad \begin{aligned} \overline{\mathbf{E}}_0 \overline{\tau}_{g_{A(T)}}^* &= B(A(T)) = \log A(T) - (1 + \mathbb{C}) + O(T^{-1} \log^2 A(T)) \\ &= \log T - (1 + \mathbb{C}) + O(T^{-1} \log^2 T). \end{aligned}$$

From (54), (55), and (51) we obtain the asymptotics (53) for large T . This completes the proof.

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