Spanning Properties of Graphs Induced by Directional Antennas

Mirela Damian

Robin Flatland

Abstract—Let S be a set of points in the plane, whose unit disk graph is connected. We address the problem of finding orientations and a minimum radius for directional antennas of a fixed cone angle placed at the points of S , such that the induced communication graph $G[S]$ is a *hop t -spanner* (meaning that $G[S]$ is *strongly connected*, and contains a directed path with at most t edges between any pair of points within unit distance). We consider problem instances in which antenna angles are fixed at 120° and 90° . We show that, in the case of 120° angles, a radius of 5 suffices to establish a hop 5-spanner; in the case of 90° angles, a radius of 7 suffices to establish a hop 6-spanner; and for any angle strictly less than 180° , a radius of 2 is necessary to establish strong connectivity.

I. INTRODUCTION

Let S be a set of points in the plane representing wireless nodes, such that the unit disk graph for S , denoted $\text{UDG}(S)$, is connected. (The unit disk graph $\text{UDG}(S)$ connects any two points in S within unit distance.) Assume that each node is equipped with one directional antenna, geometrically represented as a closed circular sector with angular aperture α and radius r (see Figure 1a). We call such an antenna an α -antenna or an (α, r) -antenna, and the region it covers we call a *wedge*.

Fig. 1. (a) Directional antenna represented as a wedge of angle α and radius r (b) ab is a directed edge in the communication graph.

An antenna orientation is specified by the counter-clockwise angle θ measured from the positive x -axis to the bisector of the wedge. The *communication graph* $G[S]$ formed by the antennas placed at points in S is a directed graph with vertex set S and edges $\vec{ab}$ directed from a to b if and only if the antenna wedge at a covers b . For a fixed integer $t > 0$, we call $G[S]$ a *hop t -spanner*, if $G[S]$ is *strongly connected*, and for each edge $ab \in \text{UDG}(S)$, there is a directed path in $G[S]$ from a to b with at most t hops (edges). In this paper we address the following problem.

Villanova University, Villanova, USA,
mirela.damian@villanova.edu.
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Siena College, Loudonville, USA, flatland@siena.edu.

Let S be a planar point set such that $\text{UDG}(S)$ is connected. For a fixed angle α , find orientations θ_i of the antennas at the points p_i in S and a minimum radius r for which the communication graph $G[S]$ is a hop t -spanner, for some integer constant $t > 0$.

We consider instances of this problem for $\alpha = 120^\circ$ (Section III) and $\alpha = 90^\circ$ (Section IV). We show that, in the case of 120° angles, a radius of 5 suffices to establish a hop 5-spanner; in the case of 90° angles, a radius of 7 suffices to establish a hop 6-spanner; and for any angle strictly less than 180° , a radius of 2 is necessary to establish strong connectivity.

A. Related Work

Caragiannis et al. [3] consider the problem of orienting α -antennas to form a strongly connected communication graph using minimal r , but they do not attempt to minimize the hops. Their technique, based on Hamiltonian tours, achieves a 3-approximation of the optimal r , but the number of hops between two nodes at unit distance can be linear. This is illustrated in Figure 2, which highlights the difference between the communication graph developed in [3], and the communication graph developed in this paper (Section III). In both cases, the antenna angles are fixed at 120° . The approach from [3] uses $r = 3$, but the induced communication graph is not a hop spanner. (For example, the unique directed path from a to b in Figure 2b has a linear number of hops.) Our approach uses a larger radius $r = 5$, but it guarantees that the induced communication graph is a hop 5-spanner. As far as we know, this is the first paper to address both connectivity and hop spanner properties of the communication graph.

In other related work, Nijnatten [4] also considers the problem of finding suitable orientations of α -antennas to form a strongly connected graph, but in his variant of the problem he allows a different r for each antenna and minimizes the overall power consumption of the network. Ben-Moshe et al. [2] consider 90° -antennas but restrict the orientations to one of the four standard quadrant directions. Bhattacharya et al. [5] consider nodes with multiple directional antennas and focus on minimizing the sum of the antenna angles for a fixed r .

Fig. 2. (a) Unit disk graph (b) communication graph from [3] (c) hop 5-spanner using 120° -antennas.

II. GENERAL ALGORITHMIC FRAMEWORK

Our algorithms for orienting 120° -antennas and 90° -antennas fall within a common algorithmic framework which we describe here. We start by finding a spanning tree of $UDG(S)$, which we call a *unit disk tree*. Specifically, we use a minimum spanning tree of $UDG(S)$ having a maximum degree of five, which can be constructed using the algorithm by Wu et al. [1]. We use the notation $MST_5(S)$ to refer to this unit disk tree. For reasons that will become clear later, we root $MST_5(S)$ at a node adjacent to at least one leaf node and at most one internal node. Such a node can be found by arbitrarily rooting the tree and then selecting any node at height one, where *height* is measured in terms of the number of edges along a longest path from the node to a leaf.

We next partition the nodes of the unit disk tree into groups as follows. For a fixed integer $k > 0$, define a k -tree as a rooted tree with k or more nodes. A k -tree is called *minimum* if none of its subtrees is a k -tree (i.e., each subtree has fewer than k nodes). For any tree T , let $N(T)$ denote the set of nodes in T . A k -

Fig. 3. (a) Partition $\Pi(T, 2)$ example (b) A $(180^\circ, 2)$ -group example

partition $\Pi(T, k)$ of a rooted tree T is a partition of the node set $N(T)$, defined recursively as follows. Starting at the leaves of T , identify a minimum k -subtree $T' \subseteq T$ whose removal does not disconnect T . The nodes of T' constitute one element of $\Pi(T, k)$. Imagine removing T' from T , then recurse on $T \setminus T'$. The process stops when fewer than k nodes remain in the tree. We refer to the node group obtained in this last recursion step as the *root group*, because it contains the root of the original tree T . The resulting partition $\Pi(T, k)$ is a collection of disjoint node groups whose union is $N(T)$. Each node

group in $\Pi(T, k)$ has k or more nodes and induces a k^2 -tree of T , with the possible exception of the root group which may contain fewer than k nodes. Figure 3a shows a 2-partition $\Pi(T, 2)$ example.

After partitioning, we determine orientations of the antennas for each node group that ensure certain local communication and coverage requirements are met. We define these requirements here in terms of what we call (α, r) -groups, relegating the details of how to meet these requirements for 120° -antennas and 90° -antennas to Sections III and IV, respectively. We say that a point $a \in S$ covers $b \in S$ if and only if a 's antenna wedge covers b .

Definition 1: A plane set of nodes I forms an (α, r) -group if there is an orientation of (α, r) -antennas at the nodes in I that guarantees (i) strong connectivity for $G[I]$, and (ii) coverage of the unit disks centered at the nodes in I .

Note that a single node in the plane is a $(360^\circ, 1)$ -group. For two nodes in the plane to form an (α, r) -group, two conditions $\alpha \geq 180^\circ$ and $r \geq 1$ are necessary. This is because an 180° -degree antenna at a single node a could cover at most half a unit disk centered at a , so a second node is necessary to cover the second half of the unit disk centered at a . This idea is depicted in Figure 3b, which shows two nodes a and b at unit distance that cooperate in covering their surrounding unit disks with $(180^\circ, 2)$ -antennas. This “unit halo” around each node in an (α, r) -group will enable us to merge two (α, r) -groups (at unit distance or less) into a single (α, r) -group, as in Lemma 1.

Lemma 1: If every node group in a partition of a rooted unit disk tree T is an (α, r) -group, then the entire node set $N(T)$ is an (α, r) -group. [Proof omitted.]

Theorem 2: For directional antennas with $\alpha < 180^\circ$, $r = 2$ is sometimes necessary to build a communication graph.

Proof: Consider a set of points positioned along the x -axis at unit intervals. For $0^\circ < \alpha < 180^\circ$, an α -antenna placed at a point p can only cover points to one side, say its left side (so the antenna at p is oriented to the left). To enable messages to flow from points left of p to points right of p , the antenna at some point left of p must be oriented to the right. The nearest such point is p 's left neighbor, which is at distance two from p 's right neighbor. Therefore, $r = 2$ is necessary. ■

III. 120° ANTENNAS

Lemma 2: For any point set I of size 3, there are orientations of 120° -antennas at the points in I such that (i) a radius equal to the maximum distance between the three points guarantees strong connectivity of $G[I]$, and (ii) for any point p in the plane, a radius equal to

the maximum distance from p to a point in I guarantees coverage of p .

Proof: Let $I = \{a, b, c\}$ be arbitrary, and assume without loss of generality that ab is a longest edge of $\triangle abc$. By the Law of Sines, $\angle acb$ is a largest angle of $\triangle abc$. Rotate I so that ab is horizontal and c lies above ab , as in Figure 4a. Reflect I across a vertical

Fig. 4. Lemma 2: Three antennas that strongly connect $\{a, b, c\}$ and cover the plane.

line if necessary, so that $\angle cab \leq \angle cba$. Then $\angle cab$ is a smallest angle of $\triangle abc$, meaning that $\angle cab \leq 60^\circ$. Orient the 120° -antenna at b so that its wedge lines up with the line segment ab and covers c . Since $|ab| \geq |ac|$, $\angle cba < 90^\circ$. This implies that b 's antenna covers a and c , establishing direct communication from b to both. Orient the 120° -antenna at c so that its wedge is a horizontal reflection of the antenna wedge at b . Then c 's antenna wedge necessarily covers a , since $\angle cab < 60^\circ$ (as established above). Finally, orient the 120° -antenna at a towards b , so that the bisector of its wedge aligns with ab . Since $\angle cab \leq 60^\circ$, a 's antenna wedge also covers c , establishing direct communication from a to both b and c . Then $G[I]$ (depicted in Figure 4b) is strongly connected, settling claim (i) of the lemma. Observe that for $r = \infty$ the three antennas collectively cover the entire plane, and any point p is contained in at least one wedge, w . Since w must still include p if r is reduced to the maximum distance from p to a point in I , this settles claim (ii) of the lemma. ■

Corollary 3: A shortest directed path between any two points in $G[I]$ (defined in Lemma 2) has at most 2 hops.

This corollary follows immediately since $G[I]$ is strongly connected and $|I| = 3$.

Lemma 3: The node set I of a minimum 3-subtree of $\text{MST}_5(S)$ is a $(120^\circ, r)$ -group, for $r = 4$. In addition, a shortest directed path between any two points in $G[I]$ has at most 4 hops.

Proof: Let T be a minimum 3-subtree of $\text{MST}_5(S)$ with node set I , and let p be the root of T . Note that the height of T must be 2 or less. Otherwise, at least one of the subtrees attached to p must have three or more nodes, contradicting the fact that T is a *minimum* 3-tree. If T has height 1, then T is topologically equivalent to the tree depicted in Figure 5a; the dashed nodes and edges may or may not exist in T . If T has height 2, then T is

topologically equivalent to one of the trees depicted in Figures 5 (b, c).

Fig. 5. Lemma 3: (a,b,c) Classes of minimum 3-trees. (d) Root group $\{p, a\}$ and communication subgraph.

Pick a *representative* node set $R = \{p, a, b\} \subseteq N(T)$ containing the root p of T . The nodes that we choose to include in R are the dark-shaded ones from Figure 5. Observe that the maximum distance between any two nodes in R is $d_{\max} = 2$. (This follows from the triangle inequality and the fact that the edges in T are no longer than 1.) We use Lemma 2(i) on R to determine orientations of the antennas at the nodes in R that strongly connect R , for $r = d_{\max}$. For each non-representative node x in T , orient the antenna at x to point towards a representative node $y \in R$. A simple analysis of the tree topologies from Figures 5(a,b,c) indicate that the maximum distance between x and y is 3, for any $x \in T \setminus R$ and any $y \in R$. In Figure 5b for instance, ax is no longer than the length of the path (a, p, b, x) , which by the triangle inequality does not exceed 3. We conclude that $r = 3$ establishes strong connectivity between all nodes of T : R is strongly connected, each node $x \in T \setminus R$ covers a node in R , and at least one of the nodes in R covers x . This latter observation follows from Lemma 2 (ii), since the maximum distance from x to a point in R is 3. Finally, a setting of $r = 4$ guarantees coverage of the unit disks centered at the nodes of T . It follows that $N(T)$ is a $(120^\circ, 4)$ -group.

Next we turn to the second claim of the lemma. So far we have established that any node $x \in T \setminus R$ can communicate directly with some node $x' \in R$; x' can communicate with any other node in R with the help of at most one intermediate node $w \in R$ (by Corollary 3); and for any node $z \in T \setminus R$, at least one node $y \in R$ can communicate directly to z . These together show that a shortest directed path from x to z in $G[I]$ contains at most 5 nodes x, x', w, y, z , and therefore at most 4 hops. ■

Theorem 4: For directional antennas with $\alpha = 120^\circ$, $r = 5$ suffices to build a hop 5-spanner $G[S]$ for a planar point set S , with $|S| \geq 3$.

Proof: We begin by constructing $T = \text{MST}_5(S)$, and a 3-partition $\Pi(T, 3)$ of T . If the root group in $\Pi(T, 3)$ consists of three or more nodes, then each node group in $\Pi(T, 3)$ is a $(120^\circ, 4)$ -group (by Lemma 3), and the union of all these groups is a $(120^\circ, 4)$ -group (by Lemma 1). This shows that $r = 4$ suffices to guarantee strong connectivity for $G[S]$.

Now we consider the case when the root group contains two nodes. (Note that a root group consisting of a single node is impossible, since we selected a root for T that is adjacent to at least one leaf node, and the 3-partition always groups the root with all adjacent leaves.) Label the two root group nodes p and a , with p the root of T and a a leaf node. Recall that the root of T is also adjacent to at most one non-leaf node and so p has at most one other child $c \neq a$ in T , which must be in a separate node group in $\Pi(T, 3)$. See Figure 5d. Orient the 120° -antenna at p so that its wedge covers a . Orient the 120° -antenna at a so that its wedge covers c . Because $|pa| \leq 1$ and $|ca| \leq 2$ (by the triangle inequality), a setting $r = 2$ enables the communication path (p, a, c) . By Lemma 3, the node set, I_c , of the 3-subtree of T rooted at c is a $(120^\circ, 4)$ -group. This implies that the representative nodes of I_c cover the unit disk centered at c , and in particular they cover p . It follows then that there is a directed cycle that starts with (p, a, c) , and (since I_c is strongly connected) continues from c to the representative node $d \in I_c$ whose wedge contains p , and then to p . This together with the strong connectivity property of the subtree rooted at c , shows that $G[S]$ is strongly connected. Note that at most 4 hops are needed to go between any two nodes in the root group: p to a is just one hop, and a to p is at most 4 hops, counted as follows: one hop from a to c , two hops from c to d (both of which are representative nodes in I_c), and one last hop from d to p . This enables us to apply the second claim of Lemma 3 to the root group.

When the root group is of size 2, the antenna orientations do not ensure the unit disks around each node are covered, as is the case for the other node groups. However, by using $r = 5$, the representative nodes of group I_c can cover these unit disks. To see this, observe that the maximum distance from a representative node of I_c to a root group node is 4, and thus any point in the unit halo of a root group node is at a maximum distance of 5. By Lemma 2 (ii), $r = 5$ is sufficient to guarantee coverage.

It remains to show that, for any $ab \in \text{UDG}(S)$, a shortest directed path in $G[S]$ from a to b has no more than 5 hops. Let I_a and I_b be the groups in $\Pi(T, 3)$ containing a and b . If I_a and I_b are identical, then a shortest directed path in $G[S]$ from a to b has no more than 4 hops (by the second claim of Lemma 3). Otherwise, consider first the cases when I_a is not the root group, or if I_a is the root group then the root group contains at least 3 nodes. Then there is an edge $\overrightarrow{a'b}$ in $G[S]$, with $a' \in I_a$. This is because the antennas placed at the nodes in I_a cover the unit halo around each node in I_a (by the first claim of Lemma 3 and Definition 1(ii)), and therefore they cover b (since $|ab| \leq 1$). A shortest directed path p in $G[S]$ from a to a' has no more than 4 hops (by the second claim of Lemma 3). This path p ,

concatenated with $\overrightarrow{a'b}$, gives us a directed path from a to b with no more than 5 hops. Now consider the case when I_a is the root group and the root group contains only 2 nodes. We again let I_c (with root c) denote the group attached to the root group. Then there is an edge $\overrightarrow{c'b}$ in $G[S]$, with $c' \in I_c$, since as described above, $r = 5$ ensures the representative nodes of I_c cover the unit disks around each root group node. A shortest directed path from a to c' has no more than 4 hops – at most two hops to get to c of I_c and then at most two more hops within I_c to get to c' , by Corollary 3 since both c and c' are representative nodes of I_c . This path, concatenated with $\overrightarrow{c'b}$, gives us a directed path from a to b with no more than 5 hops. ■

IV. 90° ANTENNAS

Due to space restrictions, we state the lemmas and theorems for 90° -antennas without proof.

Lemma 4: For any point set I of size 4, there are orientations of 90° -antennas at the points in I such that (i) a radius equal to the maximum distance between the four points guarantees strong connectivity of $G[I]$, and (ii) for any point p in the plane, a radius equal to the maximum distance from p to a point in I guarantees coverage of p .

Corollary 5: A shortest directed path between any two points in $G[I]$ (defined in Lemma 4) has at most 3 hops.

Lemma 5: The node set I of a minimum 4-subtree of $\text{MST}_5(S)$ is a $(90^\circ, r)$ -group, for $r = 6$. In addition, a shortest directed path between any two nodes in $G[I]$ has at most 5 hops.

Theorem 6: For directional antennas with $\alpha = 90^\circ$, $r = 7$ suffices to build a hop 6-spanner $G[S]$ for a planar point set S , with $|S| \geq 4$.

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