QUAL Solutions, FALL, 2004

1. Let A and B be compact sets in a metric space $\{X, d\}$. Prove that $A \cap B$ and $A \cup B$ are compact.

Solution: Since every compact set in a metric space is closed, A and B are closed, hence $A \cup B$ and $A \cap B$ are closed. Suppose that $\mathcal{F} = \{V_\alpha\}_{\alpha \in G}$ is an open cover of $A \cap B$. Since $E = (A \cap B)^c$ is open, $H = \mathcal{F} \cup \{E\}$ is an open cover of A. Since A is compact, H contains a finite subcover H' which covers A, hence covers $A \cap B$. If E is not in H' then H' is a finite subcover of $\mathcal{F}$ which covers $A \cap B$. If E is in H' (and not in $\mathcal{F}$), it can be deleted (since it contains no points of $A \cap B$) to obtain the desired subcover. Therefore $A \cap B$ is compact. Next assume that $\mathcal{K}$ is an open cover of $A \cup B$. Then $\mathcal{K}$ is an open cover of A and of B. Since A and B are both compact, $\mathcal{K}$ contains finite subcovers $\mathcal{M}$ and $\mathcal{N}$ that cover A and B, respectively. Thus $\mathcal{M} \cup \mathcal{N}$ is a finite subcover of $\mathcal{K}$ that covers $A \cup B$, and so $A \cup B$ is compact.

2. Let $\{a_n\}$ be a positive sequence. Prove that the infinite product $\prod(1 + a_n)$ is convergent if and only if the infinite sum $\sum a_n$ is convergent.

Solution: If the product is convergent, the partial products form a bounded increasing sequence. Since $a_0 + a_1 + \ldots + a_n \leq (1 + a_0)(1 + a_1) \ldots (1 + a_n)$, the partial sums are also a bounded increasing sequence, hence $\sum a_n$ is convergent. On the other hand, if the sum is convergent, its partial sums form a bounded increasing sequence. Since, for $x \geq 0$, we have $1 + x \leq \sum_{n=0}^{\infty} x^n/n! = \exp x$, by the inequality

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \leq \exp(a_1 + a_2 + \ldots + a_n),$$

the partial products also form a bounded increasing sequence, hence $\prod(1 + a_n)$ is convergent.