Let T and U be linear operators on an n-dimensional vector space V over the complex number field $\mathbb{C}$, where n is a positive integer. Suppose that T and U commute. Show that there exists a nonzero vector $v \in V$, and $\alpha, \beta \in \mathbb{C}$, such that $\mathsf{T}(v) = \alpha v$, $\mathsf{U}(v) = \beta v$, i.e., show that T and U have a common eigenvector. Hint: You may use the fact that a single linear transformation has an eigenvector.

Let V be an inner product space.

(a) Let W be a finite-dimensional subspace of the inner product space V. Prove that there exists a projection T on W along $\mathsf{W}^{\perp}$ that satisfies $\mathsf{N}(\mathsf{T}) = \mathsf{W}^{\perp}$. In addition, prove that $\|\mathsf{T}(x)\| \leq \|x\|$ for all $x \in \mathsf{V}$.

(b) If V is finite-dimensional and let U be a linear operator on V. Prove that $\mathsf{R}(\mathsf{U}^*) = \mathsf{N}(\mathsf{U})^{\perp}$. Hint: Prove that $\mathsf{R}(\mathsf{U}^*)^{\perp} = \mathsf{N}(\mathsf{U})$ first.