Qualifying Exam, Part I: Linear Algebra Problems

Unknown Date

3. If A is a real $n \times n$ matrix and n is odd, then A has at least one real eigenvalue. Prove or disprove the preceding statement.

10. Let $\mathbf{x}$ and $\mathbf{y}$ be $p \times 1$ column matrices.
Prove that $\left|\mathbf{x}^H \mathbf{y}\right| \leq \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2$.

Spring 1987

3. Let S_1 and S_2 be (not necessarily finite) linearly independent sets in a linear space V. Show that $S_1 \cup S_2$ is linearly independent and $S_1 \cap S_2 = \emptyset$ if and only if

$$[S_1] \cap [S_2] = \{0\}$$

($[S]$ denotes the linear span of S.)

7. Consider the application of Gaussian elimination *without* pivoting to a positive definite symmetric matrix A.

(i) Show that only positive pivots are encountered.

(ii) Use the results of (i) to establish the existence of the Choleski factorization $A = LL^T$, where L is lower triangular.

8. If $A = LL^T$, where L is a lower triangular matrix and A is positive definite, then complete the following algorithmic form (give reasons).

$$\begin{aligned} k &= 1, 2, \ldots, n \\ l_{kk} &= \; ? \\ i &= k+1, \ldots, n \\ l_{ik} &= \; ? \end{aligned}$$

Show that the total number of multiplications and divisions is

$$\frac{n^3}{6} + \mathrm{O}(n^2) \; .$$

Fall 1988

2. If A is an invertible linear transformation from a finite-dimensional vector space U to U, prove that there exists a polynomial p such that $A^{-1} = p(A)$.

3. Let A be an $n \times n$ matrix with the property that, for all *complex* n-vectors $\mathbf{v}$, $\mathbf{v}^h A \mathbf{v}$ is real. Prove that A is Hermitian. ($\mathbf{v}^h$ denotes the complex conjugate transpose of $\mathbf{v}$.)

5. Prove (or disprove) the following:
If A is a real $n \times n$ matrix and n is odd, then A has at least one real eigenvalue.

8. The spectral radius, $\rho(A)$, of an $n \times n$ matrix A is defined by:

$$\rho(A) = \max |\lambda_i| \ ,$$

where λ_i is an eigenvalue of A.
Show that if $\rho(A) < 1$, then $\lim_{n\to\infty} A^n = 0$.
Hint: First prove the theorem for diagonalizable matrices and use a majorization argument to extend it to the general case.

Spring 1989

1. Let $A = \mathcal{R}^n \to \mathcal{R}^n$ be a non-singular linear transformation. Show that (up to a rotation of axes) the unit sphere $\|x\| = 1$ in $\mathcal{R}^n$ is mapped by A into an ellipsoid

$$1 = \sum_{i=1}^{n} \alpha_i Y_i^2 \ , \alpha_i > 0 \ .$$

2. Let A be an $n \times n$ real matrix and λ one of its eigenvalues. Let α be the largest eigenvalue of the matrix $A^T A$. Show that

$$|\lambda| \leq \sqrt{\alpha} \ .$$

3. Prove that for any square matrix the largest sum of the moduli of the coefficients a_{ij} along any row is equal to or greater than the largest eigenvalue of the matrix.

4. Consider the linear map $A : \mathcal{R}^{n+p} \to \mathcal{R}^p$ given by

$$\sum_{j=1}^{n+p} a_{ij} x_j = 0 \quad i = 1, \dots, p$$

(that is, $Ax = 0$). What does the implicit function theorem say about the solution of these equations for the unknowns $x_{n+1}, \dots, x_{n+p}$?

9. An $n \times n$ real symmetric matrix A is said to be positive definite if there exists a positive constant C such that:

$$w^T A w \geq C w^T w$$

for all n-vectors w.
Prove that, if A is symmetric and positive definite:

(i) $a_{ii} > 0$ for all i.

(ii) The largest element of A is on the diagonal.

Hint: For (ii) you may want to prove the intermediate result: $a_{ii} a_{jj} > a_{ij}^2$ for all $i \neq j$.

10. For the system of differential equations show, obtain a new system of *first* order differential equations of the form $\dot{\xi} = A\xi$ where ξ is a vector and A is a matrix. Further, from the new system obtain an eigenvalue problem and the eigenvalue equation $f(\lambda) = 0$. (*Do not solve the last.*)

$$\begin{aligned} m_1\ddot{x}_1(t) &= -k_1x_1(t) + k_2x_2(t) - x_1(t) \\ m_2\ddot{x}_2(t) &= -k_2\left(x_2(t) - x_1(t)\right) \end{aligned}$$

m_1, m_2, k_1, k_2 are constants.

Fall 1989

3. Let A, B, and C be $n \times n$ matrices, and let λ be a number. Show that the matrix $\lambda^2 A + \lambda B + C$ is nonsingular if and only if the $2n \times 2n$ matrix

$$\lambda \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix} + \begin{pmatrix} C & B \\ 0 & -I \end{pmatrix}$$

is nonsingular.

5. Let P be a $p \times p$ matrix and let $\|\cdot\|$ denote any vector norm and its corresponding matrix norm. If $\|P\| < 1$, then prove the Banach Lemma:

(a) $I + P$ is nonsingular.

(b) $\left\|(I+P)^{-1}\right\| > 1/(1 + \|P\|)$.

(c) $1/(1 - \|P\|) > \left\|(I+P)^{-1}\right\|$.

(I denotes the $p \times p$ identity matrix.)

6. A square matrix M such that $M^p = 0$, for some positive integer p, is said to be *nilpotent.* Prove the following:

(i) The trace of a nilpotent matrix is zero.

(ii) If A and B are square matrices, then $I - AB + BA$ is not nilpotent.

7. Let A be a $n \times n$ matrix of functions $a_{ij}(x)$ which are continuous for $a \le x \le b$. Suppose that $\phi(x)$ is a matrix of functions which are continuously differentiable and that

$$\phi'(x) = A\phi\ , \quad a \le x \le b\ .$$

(ϕ' is the matrix obtained from ϕ through differentiation of every element.)
Prove that

$$\det\left(\phi(v)\right) = \det\left(\phi(u)\right) \exp \int_u^v \operatorname{tr} A(s)ds\ .$$

($\operatorname{tr} A = \text{trace } A$)

9. Consider the following equation for an unknown $n \times n$ matrix X:

$$AX - XA = B ,$$

where A and B are also $n \times n$ matrices.

Suppose that A has a complete set of right eigenvectors, $r_1, r_2, \ldots, r_n$, and a complete set of left eigenvectors, $s_1, s_2, \ldots, s_n$, with $s_i^T r_j = \delta_{ij}$.

(i) Show that the equation has a solution if and only if:

$$s_i^T B r_j = 0$$

whenever s_i and r_j correspond to the same eigenvalue.

(ii) Compute the null space, *i.e.*, all matrices Z such that

$$AZ - ZA = 0 .$$

Spring 1990

1. P is a 4×4 real matrix that can transform the unit column matrices (e_k) in the following manner, $Pe_k = e_{k+1}$ for $k = 1, 2, 3$ and $Pe_4 = e_1$.

(a) Find P, $\det(P - \lambda I)$, P^{-1}, and P^T.

(b) If we define $\mu = e^{i\pi/2}$, prove the associated eigenvectors of the complex eigenvalues $\lambda_r = \mu^{4-r}$ can be expressed as $X_r = (\mu^{1r}, \mu^{2r}, \mu^{3r}, \mu^{4r})^T$.

(c) Do these eigenvectors form an orthogonal basis? Prove it.

2. Let A be an $n \times n$ matrix.

(a) Define the minimal polynomial of A and show how it may be constructed.

(b) Prove that A is invertible if and only if the constant term of its minimal polynomial is distinct from zero.

3. Suppose that the columns of a square matrix A form an orthonormal set.

(a) Prove that the rows of A form an orthonormal set.

(b) Prove that $|\det(A)| = 1$.

4. Let V, U be linear spaces (not necessarily finite-dimensional) over the same system of scalars. Let B be a basis in V, and let $T : V \to U$ be linear. Show that T is an isomorphism if and only if TB is a basis in U.

5. A is an $m \times n$ matrix of rank r, where $r < m < n$. If $A = BC$, where B is $m \times r$ and C is $r \times n$, then show that the minimum norm least square solution of

$$Ax = b$$

is

$$x = C^T(CC^T)^{-1}(B^TB)^{-1}B^Tb \ .$$

Here x and b are, respectively, n and m dimensional column vectors and T denotes the transpose.

Fall 1990

1. For the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} ,$$

compute A^k and e^A where k is a positive, arbitrarily large number.

2. $\{X_1, X_2, \ldots, X_p\}$ is an orthonormal set of eigenvectors of a $p \times p$ matrix $B + \lambda I$ with associated eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Prove the solution of the equation $BX = b$ is

$$X = \sum_{i=1}^{p} \frac{(b, X_i)}{\lambda_i - \lambda} X_i$$

if $\lambda_i - \lambda \neq 0$ for all $1 \leq i \leq p$.

3. Let A and B be two $n \times n$ matrices over the complex numbers. show that AB and BA have the same spectrium, except possibly for the eigenvalue zero.

4. Let P_n be the set of all polynomials in t of order less than n with real coefficients. If the transformations A and D are defined, respectively, by

$$\begin{aligned} Ax(t) &= x(t+1) \ , \\ Dx(t) &= \frac{d}{dt}x(t) \ , \end{aligned}$$

for every x in P_n, prove that

$$1 + \frac{D}{1} + \frac{D^2}{2} + \cdots + \frac{D^{(n-1)}}{(n-1)} = A \ .$$

5. Let V be an inner-product space with inner-product $\langle \cdot, \cdot \rangle$ and let $X_1, X_2, \ldots, X_m \in V$. Show that the elements $X_1, X_2, \ldots, X_m$ are linearly independent if and only if $\det A \neq 0$, where A is an $m \times m$ matrix whose elements are $\langle X_i, X_j \rangle$, $i, j = 1, 2, \ldots, m$.

Spring 1991

1. Let A be a square matrix and let $a_0, a_1, \ldots, a_n$ be numbers such that

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_0 I = 0 \ .$$

Show that

(a) Any eigenvalue λ of A satisfies the equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0 I = 0 \ .$$

(b) If $a_0 \neq 0$, then A is nonsingular.

2. Let V be a finite-dimensional inner product space over $\mathcal{C}$; consider a linear map $A : V \to V$.

(a) Show that the kernel of the adjoint A^* of A is the orthogonal complement of the image of A.

(b) Show that $\lambda \in \mathcal{C}$ is an eigenvalue of A^* if and only if the image of $A - \overline{\lambda}$ is not all of V.

3. In $\mathcal{R}^3$, a vector X lies in the x_1x_2-plane. We start to rotate this vector from x_1 to x_2, first by ϕ_1 to get a vector $X^{(1)}$, and then rotate $X^{(1)}$ by ϕ_2 to get a vector $X^{(2)}$, ..., and finally to rotate $X^{(n-1)}$ by ϕ_n to get a vector $X^{(n)}$.

(a) Find a general matrix $R_{12}(\phi_i)$ that represents the rotation from $X^{(i-1)}$ by ϕ_i to get a vector $X^{(i)}$.

(b) Is $R_{12}(\phi_i)$ unitary? Prove it. Compute $\det R_{12}(\phi_i)$.

(c) Prove that the matrix representing the rotation from X to $X^{(n)}$ can be written as $R_{12}\left(\sum_{i=1}^{n} \phi_i\right)$.

4. Let P be the permutation matrix

$$\begin{pmatrix} 0 & 0 & \ldots & & 0 & 1 \\ 1 & 0 & \ldots & & 0 & 0 \\ 0 & 1 & 0 & & \ldots & 0 \\ \ldots & & & & & \\ 0 & 0 & \ldots & & 1 & 0 \end{pmatrix}$$

Find the characteristic equation and compute the eigenvalues on the unit circle of the complex plane. Explain why the eigenvectors are orthogonal. Show that if λ is an eigenvalue then $(\lambda^{n-1}, \lambda^{2(n-1)}, \ldots, \lambda^{n(n-1)})$ is the corresponding eigenvector.

5. Prove that $f(A) = 0$ if $f(\lambda)$ is the characteristic polynomial of a matrix A.

Fall 1991

1. Suppose that $Y_{ij} = \alpha_i + \delta_j$ and $\sum_{j=1}^{3} \delta_j = 0$, where Y_{ij} is a given constant and α_i and δ_j are unknown variables $i = 1, 2$ and $j = 1, 2, 3$. Prove or disprove that there is a unique solution $(\alpha_1, \delta_1, \delta_2)$.

2. Let A and B be $n \times n$ matrices. Show that the eigenvalues of the $2n \times 2n$ matrix

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

are the same as the eigenvalues of the matrices $A + B$ and $A - B$.
Hint: Let P be the matrix

$$\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

where I is the $n \times n$ identity matrix, and consider $P^{-1}MP$.

3. Prove that a square matrix A is invertible if it has a unique right inverse.

4. Let A be a real $n \times n$ matrix and v some vector in $\mathcal{R}^n$ for which $A^k v = 0$ but $A^{k-1}v$ is non-zero. Establish that

1. The vectors $A^{k-1}v, \ldots, Av, v$ span a k-dimensional subspace S.
2. A maps S into itself (S is invariant under the map A).
3. Restricting the map A to S we get a $k \times k$ matrix A_S. Find this matrix.

5. Let A be an $n \times n$ hermitian matrix and x an n-dimensional vector. Determine

$$\min_{\|x\|=1} (x, Ax) \ ,$$

the minimum of the inner product (x, Ax) on $S = \{x \ : \ \|x\| = 1\}$. Give a proof.

Spring 1992

1. Find all the possible values of x such that the matrix $\begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix}$ is positive definite.

2. Let A be an $n \times n$ matrix such that $A^2 = I$.

(i) What are the eigenvalues of A?

(ii) For each distinct eigenvalue, find an eigenvector associated with it.

(iii) What is the rank of A?

3. A basic theorem in Linear Algebra states that for any real square matrix A there is an orthogonal matrix Q and an upper triangular matrix R such that $A = QRQ^{-1}$ and the eigenvalues of A are on the diagonal of R. If A is symmetric show that R is diagonal and that the columns of Q are the eigenvectors of A.

4. Prove that the eigenvalues of a normal matrix A are all equal if and only if A is a scalar multiple of the identity matrix I.

5. The singular value decomposition for any real $m \times n$ matrix A is $A = Q_1 L Q_2^T$, where the columns of Q_1 are the eigenvectors of AA^T and the columns of Q_2 are the eigenvectors of $A^T A$. The matrix L has non-zero entries only along the first r diagonal positions and these are the square roots of the non-zero eigenvalues of AA^T.

Show that the non-zero diagonal elements of L are also the square roots of the non-zero eigenvalues of $A^T A$. Also establish that any real square A can be written as QS where Q is orthogonal and S is symmetric positive semi-definite.

Fall 1992

1. Let A be a linear transformation that maps an n-dimensional vector space V to V. Prove that A is invertible if and only if $Ax = 0$ implies $x = 0$.

2. Let A be an $n \times n$ matrix. Prove that the dimension of the row space of A equals the dimension of the column space of A.

3. Let A be an $n \times n$ real invertible matrix and x a vector in $\mathcal{R}^n$ having components x_i. Show that that is a non-singular matrix T for which the unit sphere $(x_1)^2 + \cdots + (x_n)^2 = 1$ is mapped into the ellipsoid $a_1(y_1)^2 + \cdots + a_n(y_n)^2 = 1$ under the change of variables $y = Tx$; the a_i are positive constants.

4. A matrix A is said to be *idempotent* if $A^2 = A$. Prove that the trace of a hermitian idempotent matrix is equal to its rank.

5. It is a theorem that if A is any $n \times n$ matrix there exists a unitary matrix U such that $U^{-1}AU$ is upper triangular, with the eigenvalues of A along the diagonal. Prove this for the cases $n = 2$ and $n = 3$.

Spring 1993

1. Consider the matrix

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \dots & w^{(n-1)^2} \end{pmatrix}$$

where w is a complex n^{th} root of unity: $w = \exp(2\pi i/n)$. F is a unitary matrix. Establish this for the case $n = 4$.

Hint: The sum of w^j for $j = 0, 1, \dots, n-1$ is zero (why?).

2. Let A_1, A_2, and A_3 be $m \times m$, $n \times m$, and $n \times n$ matrices, respectively. Prove that

$$\begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$$

is nonsingular if and only if both A_1 and A_3 are nonsingular.

3. Let A be a square matrix such that, for some integer $k \geq 1$ and some induced matrix norm $\|\cdot\|$, we have $\left\|A^k\right\| < 1$. If $m \geq 1$ is an integer, show that the matrix $I + A^m$ is nonsingular.

4. Let A be a Hermitian transformation on an n-dimensional vector space. Show that A has n mutually orthogonal eigenvectors (even if the eigenvalues are not distinct).

Hint: Use an induction argument.

5. Let A be a complex $n \times n$ matrix. Show that $\|Ax\|_2 = \|x\|_2$ for all complex $x \in \mathcal{C}^n$ if and only if $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all complex $x, y \in \mathcal{C}^n$ if and only if $A^H A = I$. (Here, $\langle u, v \rangle = v^H u$ and $\|u\|_2^2 = \langle u, u \rangle$.)

Fall 1993

1. For $i = 1, 2, \ldots, k$, let A_i be an $m \times n$ matrix. If, for some $1 \leq j \leq k$, rank $A_j = n$, show that the matrix $A_1^H A_1 + \cdots + A_k^H A_k$ is nonsingular.

2. Prove that the equation $x = xP$ has an infinite number of solutions x, where $x = (x_1, \ldots, x_n)$ is a row vector in $\mathcal{R}^n$, and P is an $n \times n$ matrix, for which every row sum is equal to 1.

3. Let A be an $m \times n$ matrix. Show that rank $A^H A =$ rank A. (Hint: Consider the null space of $A^H A$ and A.)

4. Let M be a nonempty subset of a (not necessarily finite-dimensional) linear space L. Show that M is linearly independent if and only if, for each $x \in M$, we have $x \notin [M - \{x\}]$. ($[N]$ denotes the linear span of the set N.)

5. Let V and U be (not necessarily finite-dimensional) linear spaces having the same system of scalars, and let B be a basis in V. If $T : V \to U$ is linear, show that T is an isomorphism if and only if TB is a basis in U.

Spring 1994

1. Let $a_1, a_2, \ldots, a_n$ form a basis in E^n. We want to replace a_j by a. Assume that a can be written in terms of the a_i's as:

$$a = \sum_{i=1}^{n} \lambda_i a_i \ .$$

Prove that the vectors $a_1, a_2, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_n$ form a basis if and only if $\lambda_j \neq 0$.

2. Let $A = (a_{i,j})$ be an $n \times n$ Hermitian matrix with eigenvalues λ_i. Define a matrix norm by

$$\|A\|^2 = \sum_{i,j=1}^{n} |a_{i,j}|^2 \ .$$

If M is the maximum of the absolute values of the eigenvalues of A, show that $\|A\| \le cM$, where c is the square root of n.

3. Let A be a square matrix and let $k \ge 1$ be an integer. Show that

$$\operatorname{rank} A = \operatorname{rank}\,[A \,\vdots\, A^2 \,\vdots\, \cdots \,\vdots\, A^k] \ .$$

4. Let A be a linear transformation on a vector space V for which $A^k u = 0$ and $A^{k-1}u$ is non-zero, for some vector u, where $k \le n$. Show that $u, Au, \ldots, A^{k-1}u$ span a k-dimensional subspace which is invariant under A.

5. Let A and B be Hermitian, positive semidefinite matrices such that $A + B$ is positive definite. If λ is a number such that $A - \lambda B$ is singular, show that λ is real and $\lambda > -1$. Moreover, using this result, show that if M is a Hermitian positive semidefinite matrix, then each eigenvalue of M is real and nonnegative. (Use the fact that kM is positive semidefinite for any $k \ge 0$.)

Fall 1994

1. Consider the polyhedral set X of the form

$$X = \{x \ : \ Ax \le b, x \ge 0\} \ ,$$

where A is $m \times n$ and b is an m-vector. The "defining hyperplanes" of X are the $m + n$ hyperplanes associated with the defining halfspaces of X. A point $x \in X$ is defined to be an *extreme point* of X if x cannot be represented as a strict convex combination of two distinct points in X.

(a) Identify all of the extreme points of $X = \{x \ : \ -2x_1 + x_2 \le 2 \le 3, x_1 \le 2, x \ge 0\}$.

(b) Prove that x is an extreme point of X if and only if x lies on some n linearly independent defining hyperplanes of X.

2. For any $n \times n$ matrix A there is a unitary matrix U and an upper triangular matrix T such that $A = UTU^{-1}$ with the eigenvalues of A on the diagonal of T. When A is Hermitian show that T is diagonal and that the columns of U are the eigenvectors of A.

3. Let A be a square matrix such that $0 \ne A \ne I$ and $A^2 = A$. Show that:

(a) A is singular.

(b) $\|A\| \ge 1$ for any induced norm $\|\cdot\|$.

4. Find the eigenvalues of the matrix

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and show that if λ is an eigenvalue of P then the corresponding eigenvector is

$$\begin{pmatrix} \lambda^{n-1} \\ \lambda^{2(n-1)} \\ \vdots \\ \lambda^{n(n-1)} \end{pmatrix} .$$

5. Let b and c be numbers, and let A and B be square matrices of the same order such that B and $A + cI$ are nonsingular. Show that:

(a) $B^{-1}AB$ and cI are nonsingular.

(b) If λ is an eigenvalue of A, then $\lambda + c \neq 0$ and $(\lambda + b)(\lambda + c)^{-1}$ is an eigenvalue of the matrix $(B^{-1}AB + bI)(B^{-1}AB + cI)^{-1}$.

Spring 1995

1. Let Q be an $n \times n$ matrix with matrix norm $\|Q\| < 1$. Show that the equation $u = Qu + b$ has a unique solution for any given n-vector b.

2. Let B be an $m \times n$ matrix, where $m < n$, and B has full row rank (*i.e.*, the rows of B are linearly independent). Show that the matrix BB^t is invertible.

Comment: This fact is central to both Karmarkar's algorithm and the Ellipsoid algorithm for Linear Programming.

3. Let A, B, and C be Hermitian, positive definite $n \times n$ matrices such that for some scalar k we have

$$\det(k^2 A + kB + C) = 0 .$$

Show that Re $k < 0$.

4. Let $u_1, \dots, u_n$ be an orthogonal basis for $\mathcal{R}^n$ with $u_1, \dots, u_k$ in the null space of an $m \times n$ matrix A. We suppose $k < n$. Prove the theorem that the dimension of the range of A (namely, the column space of A) plus the dimension of the null space of A is n.

5. Let a square matrix A be partitioned into blocks A_{ij}, $i, j = 1, 2, \dots, k$, such that each A_{ii} is a square matrix, $i = 1, 2, \dots, k$, and $A_{ij} = 0$, $1 \leq i < j \leq k$. Show that A is nonsingular if and only if each matrix A_{ii} is nonsingular, $i = 1, 2, \dots, k$.

(Hint: First prove the assertion for $k = 2$ and then deduce the claim for arbitrary $k > 2$.)

Fall 1995

1. Let $u_1, \ldots, u_n$ be a basis for a vector space V and let T be a linear transformation on V for which $T(u_1) = 0$, $T(u_2) = u_1$, $T(u_3) = u_1 + u_2$, $\ldots$, $T(u_n) = u_1 + \ldots + u_{n-1}$.
Show that $T^n = 0$. Hint: Use an induction argument.

2. Consider the polyhedral set X of the form

$$X = \{x \ : \ Ax \leq b\} ,$$

where A is $m \times n$ and b is an m-vector. The "defining hyperplanes" of X are the m hyperplanes associated with the defining halfspaces of X. A point $x \in X$ is defined to be an *extreme point* of X if x cannot be represented as a strict convex combination of two distinct points in X.

(a) Identify all of the extreme points of $X = \{x \ : \ -3x_1 + x_2 \leq -2, -x_1 + x_2 \leq 2, -x_1 + 2x_2 \leq 8, -x_2 \leq -2, x \geq 0\}$.

(b) Recall that a *line* is an affine space of dimension 1. Prove that the polyhedral set X has extreme points if and only if it contains no lines. (X contains a line if the entire line is a subset of X.)

3. Let A be a hermitian, positive definite matrix. Prove that

(a) A is nonsingular.

(b) A^k is a hermitian, positive definite matrix for any integer k ($A^k = (A^{-1})^{-k}$ for any $k < 0$, and $A^0 = I$).

4. If S is a subspace of a vector space V we know that V is the direct sum of S and the orthogonal complement of S (we designate this complement by S^+).
Prove that $S = (S^+)^+$.

5. Let A and B be hermitian, positive definite matrices of the same order. Prove that

(a) A and $A + B$ are nonsingular.

(b) $A^{-1} - (A + B)^{-1}$ is a hermitian, positive definite matrix. (Hint: Use the identity $M^{-1} - N^{-1} = M^{-1}(N - M)N^{-1}$.)

Spring 1996

1. Consider a nonempty set $X = \{x \in \mathcal{R}^n \ : \ Ax = b, x \geq 0\}$. (Here, A is an $m \times n$ matrix, $x \in \mathcal{R}^n$ is a column vector, and $b \in \mathcal{R}^m$ is a column vector.) We say that $d \in \mathcal{R}^n$ is a *direction* of the set X if for each $x_0 \in X$, the ray $\{x_0 + \lambda d \ : \ \lambda \geq 0, \lambda \in \mathcal{R}^1\}$ is contained in the set X.
Show that d is a direction of X if and only if $d \neq 0$, $Ad = 0$, and $d \geq 0$.

2. Let S be a k-dimensional subspace of a vector space V and $u_1, u_2, \ldots, u_k$ an orthogonal basis of S. If v is any element of V, show that v can be written as a sum of v_1 and v_2, where v_1 lies in S and v_2 belongs to the orthogonal complement of S.

3. Let A be a Hermitian, positive definite [positive semi-definite] matrix, and let $k \geq 1$ be an integer. Show that there exists a Hermitian, positive definite [positive semi-definite] matrix B such that $B^k = A$.

4. Let A be an $m \times n$ matrix and suppose that y is in $\mathcal{R}^m$. Show that either $Ax = y$ for some vector x in $\mathcal{R}^n$ or there is some w in $\mathcal{R}^m$ for which $A^T w = 0$ and the inner product between y and w is non-zero.

If y_1 is the orthogonal projection of y onto the range of A, show that $y - y_1$ belongs to the orthogonal complement of the null space of A^T.

5. Let A be a Hermitian, positive definite matrix. Show that:

(i) A^{-1} exists.

(ii) There exists a Hermitian, positive definite matrix B such that $B^2 = A$.

(iii) The matrix $M = A + A^{-1} - 2I$ is Hermitian, positive semidefinite. If, in addition, 1 is not an eigenvalue of A, then M is positive definite.

Hints:

(ii) Use the eigenvalue decomposition theorem.

(iii) Consider the matrix $(B - B^{-1})^2$ with $B^2 = A$.

Fall 1996

1. For any set $X \subseteq \mathcal{R}^n$, a point $\mathbf{x} \in X$ is defined to be an *extreme point* of X if $\mathbf{x}$ cannot be represented as a strict convex combination of two distinct points in X. (We say that $\mathbf{x}$ is a strict convex combination of $\mathbf{y}$ and $\mathbf{z}$ if there exists a real number, $\mu \in \mathcal{R}^1$, such that $\mathbf{x} = \mu\mathbf{y} + (1-\mu)\mathbf{z}$, with $0 < \mu < 1$)

A *convex cone* $C \subseteq \mathcal{R}^n$ is a convex set with the additional property that $\lambda\mathbf{x} \in C$ for each $\mathbf{x} \in C$ and for each $\lambda \geq 0$.

(a) Show that if C is a convex cone, then C has at most one extreme point. (What must the extreme point be, if there is one?)

(b) Give an example of a nonempty convex cone that has no extreme points.

2. A norm $\|A\|$ can be defined for any square matrix. It has the properties that $\|A\|$ is nonnegative (and zero only when A is zero); for any scalar r, $\|rA\| = |r|\,\|A\|$, and $\|A + B\| \leq \|A\| + \|B\|$, if B is another matrix of the same size. Show that if $\|A\| \leq 1$ then $I - A$ has an inverse. Using this establish that if $\|A - B\| < \delta$, for some sufficiently small δ, then B is also invertible.

3. For two $n \times n$ matrices A and B such that $AB = BA$, prove that the null space of A is an invariant subspace of B. Prove further that if x is an eigenvector of A with multiplicity one, then x is also an eigenvector of B.

4. Find the eigenvalues of the matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

and show that if λ is an eigenvalue of P then the corresponding eigenvector is

$$\begin{pmatrix} \lambda^{n-1} \\ \lambda^{2(n-1)} \\ \vdots \\ \lambda^{n(n-1)} \end{pmatrix} .$$

5. The vectors $(1, 2, 2)$, $(-1, 0, 1)$, and $(2, 1, 0)$ form a basis for $\mathcal{R}^3$. From these vectors construct an orthonormal basis of $\mathcal{R}^3$. Use this orthonormal basis to find a vector x in $\mathcal{R}^3$ that best approximates $z = (1, -2, 1)$ in the sense that $\|x - z\|_2$ is a minimum.

Spring 1997

1. Define a norm by $\|A\| = \sqrt{\text{trace}(A^T A)}$, for any $n \times n$ symmetric matrix A. If M denotes the largest magnitude of the eigenvalues of A, show that $\|A\| \le \sqrt{n}M$.

2. For a convex set $X \subseteq \mathcal{R}^n$, a nonzero vector $\mathbf{d} \in \mathcal{R}^n$ is defined to be a *direction* of X if, for each $\mathbf{x}_0 \in X$, the ray $\{\mathbf{x}_0 + \lambda\mathbf{d} \ : \ \lambda \ge 0\}$ is a subset of the set X.

(a) Consider the set $X = \{(x_1, x_2) \ : \ x_1 - 2x_2 \ge -6, x_1 - x_2 \ge -2, x_1 \ge 0, x_2 \ge 1\} \subset \mathcal{R}^2$. Describe clearly in words, or using analytic geometry, what the set of directions is for X. Give one explicit direction vector, $\mathbf{d}$.

(b) Prove that $\mathbf{d}$ is a direction of $X = \{\mathbf{x} \ : \ \mathbf{Ax} \le b, \mathbf{x} \ge 0\}$ if and only if $\mathbf{d} \ge 0$, $\mathbf{d} \ne 0$, and $\mathbf{Ad} \le 0$.

3. Let A be a symmetric (also known as "real hermitian") $n \times n$ matrix.

(a) Define a function $f(X)$ for $X \in \mathcal{R}^n$ as $f(X) = AX \cdot X$. If we restrict f to those vectors X on the unit sphere ($X \cdot X = 1$), f has a maximum value. Show that this maximum value is an eigenvalue of A, and occurs at an eigenvector of A. (Hint: Lagrange multipliers)

(b) Use the result from (a) to show that A is diagonalizable. (Hint: Get "them" one by one.)

4. The Singular Value Decomposition of a real $m \times n$ matrix A is $Q_1 L (Q_2)^T$, where the columns of Q_1 are the eigenvectors of AA^T, the columns of Q_2 are eigenvectors of $A^T A$, and L is a diagonal matrix whose non-zero entries are equal to the squares of the non-zero eigenvalues of A. With this information, show that if A is square then it can be written as QS, where Q is an orthogonal matrix and S is positive-semidefinite and symmetric.

5.

$$\text{Let } B = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \dots & w^{(n-1)^2} \end{pmatrix}$$

where w is an n^{th} root of unity: $w = \exp(2\pi i/n)$. Show that B is a *unitary* matrix, namely $B^H B = I$, where H denotes complex conjugate transpose.

Fall 1997

1. Let B be an $m \times n$ matrix, where $m \leq n$, and B has full row rank (*i.e.*, the rows of B are linearly independent). Show that the matrix BB^t is invertible.

2. Denote by M^n the vector space of $n \times n$ matrices.

(a) For any $A \in M^n$, we define

$$L_A = \{B \in M^n \ : \ AB = BA\} \ .$$

Show that L_A is a subspace of M^n of dimension at least equal to n.

(b) Let $n = 2$, and $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. Determine a basis for L_A.

3. Let A be a square matrix and suppose A is partitioned in the form of

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \ .$$

If A_{22} is a square non-singular matrix, show that

$$|A| = |A_{22}| \left| A_{11} - A_{12} A_{22}^{-1} A_{21} \right| \ .$$

4. Let S be a subspace of $\mathcal{R}^n$ and, for any y in $\mathcal{R}^n$, let z be the orthogonal projection of y onto S. Show that for all v in S distinct from z one has

$$\|y - z\| < \|y - v\| \ .$$

5. An $n \times n$ symmetric matrix A has orthonormal eigenvectors $u_1, \dots, u_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$. Show that

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T \ .$$

Spring 1998

1. Let A be a real 2×2 matrix with negative determinant. Prove that there exists a real nonsingular matrix P such that $P^{-1}AP$ is diagonal.

2. Let $v_1, v_2, \ldots, v_m$ be a basis for a vector space V and S a linear transformation on V such that $S(v_1) = 0$, and $S(v_k) = v_1 + v_2 + \cdots + v_{k-1}$, for $k = 2, \ldots, m$. Prove that $S^m = 0$.

3. Let V be a finite-dimensional inner product space, and let W be a subspace of V. Then $V = W \oplus W^\perp$, that is, each α in V is uniquely expressible in the form $\alpha = \beta + \gamma$, where $\beta \in W$ and $\gamma \in W^\perp$ ($W^\perp$ denotes the orthogonal complement of W). Define a linear operator $U : V \to V$ by $U\alpha = \beta - \gamma$.

(a) Prove that U is both self-adjoint and unitary.

(b) If $V = \mathcal{R}^3$ with the standard inner product and W is the subspace spanned by $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, find the matrix of U in the standard ordered basis.

4. Let $\mathbf{h}$ be a constant $n \times 1$ vector and $\mathbf{B}$ be an $n \times n$ real symmetric positive definite matrix.

(a) Show that

$$\sup_{\mathbf{x} \neq 0} \frac{(\mathbf{x}^T\mathbf{h})^2}{\mathbf{x}^T\mathbf{B}\mathbf{x}} = \sup_{\mathbf{u} \neq 0} \frac{(\mathbf{u}^T\mathbf{a})^2}{\mathbf{u}^T\mathbf{u}} = \mathbf{h}^T\mathbf{B}^{-1}\mathbf{h}$$

for some $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{a}$. (Hint: You may need to use the Cauchy-Schwarz inequality.)

(b) Show that the sup in (a) is attained when $\mathbf{x} = \lambda\mathbf{B}^{-1}\mathbf{h}$, for a scalar $\lambda \neq 0$.

5. Prove that the equation $u = Bu + c$ has a unique solution u^* in $\mathcal{R}^n$, where B is an $n \times n$ real matrix with $\|B\| < 1$ and c is a given vector in $\mathcal{R}^n$. Using this result, establish that the iterates $u_{n+1} = Bu_n + c$ converge in norm to u^* as n goes to infinity, for any initial u_0.

Fall 1998

1. An invertible matrix is partitioned into block form as

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} and A_{22} are square matrices.
Find a formula for A^{-1} in terms of A_{11}, A_{21}, and A_{22}.

2. Let T be a linear operator on a (not necessarily finite-dimensional) vector space V. Suppose that every subspace of V is invariant under T. Prove that T is a scalar multiple of the identity operator.

3. An $n \times n$ matrix B has a null space $N(B)$ of dimension $r < n$ (the null space consists of all solutions to the equations $Bx = 0$). The range of B, called $R(B)$, is the set of all vectors b for which there is a solution x in the system of equations $Bx = b$. Show that the dimension of the vector space $R(B)$ is $n - r$.

4. Let U be a linear operator on a finite-dimensional inner product space V.
Prove or give a counterexample: If $\|Ux\| = \|x\|$ for all vectors x in some orthonormal basis for V, then U is unitary.

5. Let $\mathbf{B}$ be an $n \times n$ real symmetric matrix and $\mathbf{P}$ be an orthogonal matrix such that $\mathbf{P}^T\mathbf{B}\mathbf{P} = \mathbf{D} = \text{diag}(d_1, d_2, \ldots, d_n)$. Prove that

(a) $\text{trace}(\mathbf{B}^r) = \sum_{i=1}^n d_i^r$ for any positive integer r.

(b) for θ in a neighborhood of zero, $|\mathbf{I} - \theta\mathbf{B}| = \prod_{i=1}^n (1 - \theta d_i)$, where $\mathbf{I}$ is the n dimensional identity matrix and $|\mathbf{A}|$ is the determinant of a square matrix $\mathbf{A}$.

(c)

$$\log|\mathbf{I} - \theta\mathbf{B}| = -\sum_{r=1}^{\infty} \frac{\text{trace}\left[(\theta\mathbf{B})^r\right]}{r}\ .$$

Spring 1999

1. If S is a subspace of a finite-dimensional inner product space V, we know that V is the direct sum of S and the orthogonal complement $S^\perp$. Prove that $S = (S^\perp)^\perp$.

2. Prove that if u and v are $m \times 1$ vectors, $\det(I + uv^t) = 1 + v^t u$.

3. Let V be a finite-dimensional vector space, T a linear transformation on V, and x a nonzero vector in V.

Show that a basis for the subspace $W = \text{span}\,\{x, Tx, T^2x, T^3x, \ldots\}$ is $\{x, Tx, T^2x, \ldots, T^{k-1}x\}$, where $k = \dim W$.

4. Let X be a real $n \times k$ matrix with rank k, $n > k$. Define $P = X(X^tX)^{-1}X^t$.

(a) Show that P and $I_n - P$ are both symmetric and idempotent, where I_n is the $n \times n$ identity matrix.

(b) Show that the set of eigenvalues of $I_n - P$ consists of $n - k$ 1's and k 0's.

5. Let T be a linear transformation on a finite-dimensional vector space V, and let $p(x)$ be the characteristic polynomial of T. Suppose that $p(x)$ can be written in the form

$$p(x) = (x - \lambda)^m q(x)\,,$$

where m is a nonnegative integer and $q(x)$ is a polynomial for which $q(\lambda) \neq 0$.

Prove that the dimension of the null space (kernel) of $T - \lambda I$ is less than or equal to m.