Doctoral Qualifying Examination – Fall 2004
Linear Algebra and Advanced Calculus

DIRECTIONS: Answer 6 questions, 3 from Linear Algebra, 3 from Advanced Calculus. Good luck!

**Linear Algebra:**

1. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

   (a) Find the dimension and a basis for the null space of $\mathbf{A}$.
   (b) Determine whether the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution, no solution, or multiple solutions. Justify your answer.

2. Let $\mathbf{V}$ be the vector space over $\mathbf{R}$ of polynomials of degree $\leq 2$.
   (a) Show that $\langle p, q\rangle \equiv p(-1)q(-1) + p(0)q(0) + p(1)q(1)$ defines an inner product in $\mathbf{V}$. [Recall: A nonzero polynomial of degree $n$ has at most $n$ distinct real zeros.]
   (b) Find an orthogonal basis for $\mathbf{V}$.

3. Let $\mathbf{A}$ be a real, skew-symmetric $n \times n$ matrix (i.e., $\mathbf{A}^T = -\mathbf{A}$). Show that all the eigenvalues of $\mathbf{A}$ are purely imaginary (i.e., if $\lambda$ is an eigenvalue, then $\lambda = i\mu$ with $\mu$ real).

4. Let $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.
   (a) Find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{DP}$ is diagonal.
   (b) Find $f(\mathbf{A})$, where $f(x) = x^4 - 3x^3 - 6x^2 + 7x$.
   (c) Find a "positive square root" of $\mathbf{A}$, that is, a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$ and $\mathbf{B}$ has positive eigenvalues.

**Advanced Calculus:**

1. Let $f(x) = \sum_{n=1}^{\infty} \sin^2(\sqrt{n}\,x)/n^2$.
   (a) Prove that $f(x)$ is continuous for all $x$.
   (b) Evaluate $\lim_{x\to 0} f(x)$.

2. Is the improper integral $\int_0^{\infty} \alpha e^{-\alpha x}\,dx$ uniformly convergent for $\alpha \geq 0$? Justify your answer. (Hint: Consider the limits: $\lim_{\alpha\to 0^+} \int_0^{\infty} \alpha e^{-\alpha x}\,dx$ and $\int_0^{\infty} \lim_{\alpha\to 0^+} (\alpha e^{-\alpha x})\,dx$.)

3. Investigate the convergence of the following series. In case of convergence, test for absolute or conditional convergence. Please do not forget to justify your answers.

   (a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-3}{2}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,5^n}$

4. Let $u_1 = 3$ and $u_{n+1} = \sqrt{u_n + 1}$, $n = 1, 2, \ldots$.
   (a) Prove that the sequence $\{u_n\}$ converges.
   (b) Evaluate $\lim_{n\to\infty} u_n$.