Problems from AMS 504 for Qualify Exam

1. Let f, g: $[0,1] \to [0,\infty)$ be continuous functions satisfying

$$\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x)$$

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Prove that there exists $t \in [0,1]$ with $f^2(t) + 3f(t) = g^2(t) + 3g(t)$.

2. Let $f : R \to R$ be twice differentiable, and suppose that for all $x \in R$, $|f(x)| \le 1$ and $|f''(x)| \le 1$. Prove that $|f'(x)| \le 2$ for all $x \in R$.