Common Qualifying Exam Part B

Computational and Applied Math

Solve three out of four problems

(1) Find the radius of convergence of power series:

(a).

$$\sum \frac{(-1)^n n! x^n}{n^n} z^n i$$

(b).

$$\sum \frac{n}{2^{n+1}} z^n.$$

(2) Are closures and interiors of connected sets always connected? If yes, prove your statement; if no, show an example.

(3) If A is an $n \times n$ diagonalizable matrix and the spectral radius is defined as the maximum magnitude of eigenvalues of A, that is

$$\rho(A) = |\lambda_i|_{max}$$

where λ_i is the i-th eigenvalue of A.

1. Show that if $\rho(A) < 1$, then $\lim_{n\to\infty} A^n = 0$.

(a). Show that the vector sequence $x_{k+1} = Ax_k + b$ is convergent if the spectral radius of A is less than 1.

(b). Show that if the spectral radius is less than 1, then

$$I + A + A^2 + A^3 + \cdots = (I - A)^{-1},$$

but not otherwise.

(4) For the following matrix

$$A = \begin{pmatrix} 7 & 1 & 0 & 1 \\ 1 & 6 & 1 & 1 \\ 0 & 1 & 8 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Give the upper bound M and lower bound m of all eigenvalues such that

$$M \geq |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq m$$