# AMS Common Exam - Part A, January 2009

Name: ________________________________ ID Num. ____________________

Part A: _____ / 75

Part B: _____ / 75 Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

**Good Luck!**

## Section 1: Linear Algebra

**Choose three of the four problems to solve.**

1. Consider the linear system of real variables:

$$\begin{array}{rcrcrcl} kx & + & y & + & 2z & = & 0 \\ kz & + & ky & + & z & = & p \\ k^2z & + & k^2y & + & 2kz & = & q \end{array}$$

   (a) For what values of $k, p, q \in \mathbb{R}$ will the system have (i) one, (ii) many, of (iii) no solution?

   (b) When there is a unique solution, what is it?

   (c) When there are many solutions, what is the general solution in parametric form? What is the dimension and a basis for the associated homogenous system?

2. Consider the linear map $F : \mathbb{R}^5 \to \mathbb{R}^3$:

$$F(r, s, t, u, v) = (r + 2s + 2u + 3v, r + 3s - t + 2u + 5v, 2r + 3s + 2t + 5u + 3v)$$

   (a) What is the dimension of and a basis for both the kernel and the image of $F$?

   (b) If possible, find the inverse mapping, $F^{-1}$, or explain why this is not possible.

3. Consider the quadratic matrix expression:

$$\mathbf{A}\mathbf{X}^2 + \mathbf{B}\mathbf{X} + \mathbf{C} = \mathbf{0}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{X}$ are all real, cummuting matrices.

(a) Find a general solution for $\mathbf{X}$, in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Show all steps leading to your result. What properties must $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have for the solution to exist?

(b) Can this result be extended to non-commuting matrices? Clearly explain why or why not.

4. Consider the linear transformation, $F : \mathbb{R}^3 \to \mathbb{R}^3$:

$$F(x, y, z) = (y, 3y - x + z, -y)$$

(a) Find a matrix $\mathbf{A}$ such that:

$$\vec{F}(\vec{u}) = \mathbf{A}\vec{u}$$

for any vector $\vec{u} = (x, y, z)^T$. ($\vec{F}(\vec{u})$ and $\vec{u}$ are column vectors, relative to the usual basis).

(b) Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, or explain why this is not possible.

(c) For what values of $k \in \mathbb{R}$ is the expression $\mathbf{B} = \mathbf{A}^k$ well-defined? What is a general expression for $\mathbf{A}^k$?

## Section 2: Advanced Calculus

**Choose three of the four problems to solve.**

1. Consider the following rational function:

$$g(x) = \frac{2x^2 + 3x - 2}{x^2 - 1}$$

Sketch the curve of $y = g(x)$, clearly showing:

(a) All roots, relative maxima and minima, and inflection points.

(b) Any points of discontinuity and the limits of the function in the neighborhood of these points.

(c) The limits of $g(x)$ as $x$ approaches $\pm\infty$.

2. Consider the definite integral of a function of two real variables:

$$\int\int_R \frac{(x^2+y^2)e^{(x^2-y^2)}}{2+xy}dxdy$$

where the region $R$ is defined by:

$$\frac{-1}{x} < y < \frac{1}{x}; \qquad 1 < x^2 - y^2 < 9; \qquad x \geq 0$$

(a) Sketch $R$, and find a coordinate substitution that transforms $R$ into a rectangular region.

(b) Evaluate the given integral.

*Note: The appropriate substitution is* **not** *polar or elliptical coordinates.*

3. The binomial theorem gives an expansion for the sum of two real values raised to the power of a positive integer:

$$(a+b)^n = \sum_{k=0}^{n} {}_nC_k a^{n-k} b^k \qquad \forall a, b \in \mathbb{R}, \forall n \in \mathbb{N}$$

where ${}_nC_k = \frac{n!}{k!(n-k)!}$.

(a) Prove this statement, using mathematical induction (note $0! = 1$).

4. Consider the following expression:

$$dw = \left( x\cos x + \frac{xy}{z} + 2xe^{x^2+z^2} \right) dx + \left( \frac{x^2}{2z} + \frac{1}{y^2-1} \right) dy + \left( 2ze^{x^2+z^2} - \frac{x^2 y}{2z^2} \right) dz$$

(a) Demonstrate (using differentiation), whether or not $dw$ is an exact differential.

(b) If possible, find a function $w(x, y, z)$ for which the differential is $dw$, or explain why this is not possible.