1. Let A be a real symmetric $n \times n-$matrix with n distinct eigenvalues. Let B be another $n \times n-$matrix, which commutes with A, i.e. $AB = BA$ following the matrix multiplication. Prove that $B = F(A)$, where $F(x)$ represents a polynomial in x.

2. (a) For a matrix A, the exponential function of the matrix A is a matrix defined formally by the infinite series:

$$exp(A) = e^A = I + \sum_1^\infty \frac{A^n}{n!}.$$

Prove that e^A is always nonsingular.

(b) Let A be a $n \times n-$matrix defined as follows: $a_{k,k} = 2, k = 1, 2, \cdots, n; a_{1,n} = 1; a_{k-1,k} = 1, k = 2, 3, \cdots n; a_{j,k} = 0$(the remaining entries are zero.)

$$\begin{pmatrix} 2 & 0 & 0 & \dots & 0 & 1 \\ 1 & 2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 2 & 0 \\ 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

Calculate determinant of A. (The principal diagonal consists of 2's and the diagonal immediately below has 1. The rightmost entry in the first row is also 1.)

3. Let V denote a finite-dimensional linear vector space with an inner product (scalar product) and let $T : V \rightarrow V$ be a linear operator. Let T^* denote the adjoint of T defined by the relation $(Tv_1, v_2) = (v_1, T^*v_2)$. Prove the following:

$$R(T^*) = N(T)^\perp, N(T) = R(T^*)^\perp.$$