AMS Common Exam - Part A, January 2010

Name: ________________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Prove each of the following equalities for arbitrary n-square matrices, $\mathbf{A}$ and $\mathbf{B}$, related by elementary row operations:

   (a) If $\mathbf{B}$ is related to $\mathbf{A}$ by the exchange of any two rows ($R_i \leftrightarrow R_j$):

   $$|\mathbf{B}| = -|\mathbf{A}|$$

   (b) If $\mathbf{B}$ is related to $\mathbf{A}$ by the multiplication of any row by a scalar, k ($kR_i \rightarrow R_i$):

   $$|\mathbf{B}| = k|\mathbf{A}|$$

   (c) If $\mathbf{B}$ is related to $\mathbf{A}$ by addition of the scalar multiple of any row to any other row ($kR_i + R_j \rightarrow R_j$):

   $$|\mathbf{B}| = |\mathbf{A}|$$

2. Consider the linear mapping from $\mathbb{R}^{2\times 2} \to \mathbb{P}^2$, defined by:

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - b + c + d) + (a + 2c - d)t + (a + b + 3c - 3d)t^2$$

(a) Find a basis for, and the dimension of, both the image and the kernel of F.

(b) Classify the mapping as one-to-one, onto, both, or neither, and comment on how this relates to the invertibility of the mapping.

3. Consider two block matrices, $\mathbf{U}$ with $M \times P$ blocks and $\mathbf{V}$ with $P \times N$ blocks (*i.e.* the number of vertical blocks of U is equal to the number of horizontal blocks of V):

$$\mathbf{U} = \left[\mathbf{U}_{IK}\right], \quad I \leq M, \quad K \leq P, \quad I, K, M, P \in \mathbb{N}$$

$$\mathbf{V} = \left[\mathbf{V}_{KJ}\right], \quad K \leq P, \quad J \leq N, \quad K, J, P, N \in \mathbb{N}$$

Further consider that the number of columns in any block $\mathbf{U}_{IK}$ is equal to the number of rows in any block $\mathbf{V}_{KJ}$:

$$\mathbf{U}_{IK} = [u_{IK,ik}], \quad i \leq m_I, \quad k \leq p_K, \quad i, k, m_I, p_K \in \mathbb{N}$$

$$\mathbf{V}_{KJ} = [v_{KJ,kj}], \quad k \leq p_K, \quad j \leq n_J, \quad k, j, p_K, j_N \in \mathbb{N}$$

Show that the product $\mathbf{W} = \mathbf{U}\mathbf{V}$ is an $M \times N$ block matrix, $\mathbf{W} = [\mathbf{W}_{IJ}]$, where each block is defined by:

$$\mathbf{W}_{IJ} = \sum_{K=1}^{P} \mathbf{U}_{IK}\mathbf{V}_{KJ}$$

4. Consider the matrix:

$$\mathbf{Q} = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$$

(a) If possible, find a diagonal matrix similar to $\mathbf{Q}$, or explain why this is not possible.

(b) If possible, find the inverse of $\mathbf{Q}$, or explain why this is not possible.

Note: In both cases, be sure to explain your response clearly and in detail.

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. The two surfaces defined by:

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$$

and:

$$z = x + y$$

intersect in a space curve.

(a) Find the maximum and minimum distances from the origin to this curve.

(b) Comment on the geometric interpretation of these results.

2. Consider an abitrary, continuously differentiable function of two variables, $V(x, y)$, and the coordinate substitution $x = \rho\cos\phi$, $y = \rho\sin\phi$.

   (a) Prove that there is no functional relationalship between ρ and ϕ; that is, there is no function g such that $g(\rho, \phi) = 0$ for all ρ and ϕ.

   (b) Show that $\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial \phi}\right)^2$

3. Consider a solid (uniform density) sphere of radius a, with two cylindrical holes of radius $b < a$ drilled such that both pass through the center of the sphere and are orthogonal to one another. Find the volume of the remaining solid.

4. Show that:

$$\int_0^1 \frac{x}{(x+1)^2(x^2+1)}dx = \frac{\pi - 2}{8}$$