AMS Common Exam - Part A, January 2011

Name: ________________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider the subspace of R^4 defined by $V = span(S)$, where $S = \{(1,2,1,2),(3,6,k,3),(4,8,5,9)\}$.

    (a). For what value of k, $dim(V) \neq 3$?

    (b). For the value of k in (a), find a system whose solution space is S.

    (c). If k=2, find a vector which is orthogonal with the three basis in S.

2. (a). Let $A, B \in R^{n\times n}$. $I - AB$ is invertible. Prove or disprove that $I - BA$ is invertible.

   (b). Let $A, B \in R^{n\times n}$. A, B, and $I - AB$ are invertible. Prove or disprove $(A - B^{-1})^{-1} - A^{-1}$ is invertible.

3. $\begin{cases} 2x_1 + \lambda x_2 - x_3 = 1 \\ \lambda x_1 - x_2 + x_3 = 2 \\ 4x_1 + 5x_2 - 5x_3 = -1 \end{cases}$, what is λ, when system has

(a). no solution;

(b). a unique solution;

(c). infinite number of solutions.

4. In $\boldsymbol{R}^4$, there are two sets of basis $\alpha_1, \cdots, \alpha_4$ and $\beta_1, \cdots, \beta_4$. If there is a vector whose coordinate is (a, b, c, d) in basis β, then what is its coordinate in basis α?

$$\begin{cases} \alpha_1 = (1,1,1,1) \\ \alpha_2 = (1,1,-1,-1) \\ \alpha_3 = (1,-1,1,-1) \\ \alpha_4 = (1,-1,-1,1) \end{cases}, \quad \begin{cases} \beta_1 = (1,1,0,1) \\ \beta_2 = (2,1,3,1) \\ \beta_3 = (1,1,0,0) \\ \beta_4 = (0,1,-1,-1) \end{cases}.$$

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Solve the following two problems

   (a). If

$$f(x) = \begin{cases} x^5 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases},$$

   what is the highest order of derivative that exists at point $x = 0$?

   (b). If $0 < a < b$, use mean value theorem to prove $1 - \dfrac{a}{b} < \ln \dfrac{b}{a} < \dfrac{b}{a} - 1$.

2. Evaluate $\oint_L \dfrac{(x+y)dx-(x-y)dy}{x^2+y^2}$, where L is counterclockwise with expression

(a). $(x-2)^2+y^2=1$;

(b). $x^2+y^2=1$;

(c). any simple closed curve which encircles the point of origin.

3. Evaluate $\iint\limits_G e^{\frac{y}{x+y}}\, dxdy$, where G is the triangle enclosed by line $x + y = 1$, x axis and y axis.

4. If $a > 0$, $b > 0$, evaluate the area enclosed by curve $\sqrt[4]{\frac{x}{a}} + \sqrt[4]{\frac{y}{b}} = 1$, $x = 0$, $y = 0$.

(Hint: you can use coordinate transformation, where $x = ar\cos^{\alpha}\psi$, $y = br\sin^{\alpha}\psi$, and choose a proper value for α.)