

AMS Common Exam - Part A, June 2010

Name: _____

ID Num. _____

Part A: _____ / 75

Part B: _____ / 75

Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider the set of three vectors from \mathbb{R}^3 :

$$T = \{(2, 1, -1), (1, 0, 1), (0, 1, -1)\}$$

- (a) Prove that the set, T , forms a basis for \mathbb{R}^3 .
- (b) Express the vectors of the usual basis, $(1, 0, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ as coordinates relative to the basis T .
- (c) Find the matrix that represents this change of basis.

2. Given the linear transformation $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

$$F(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$$

- (a) Find a matrix representation, \mathbf{A} , of F .
- (b) Find all eigenvalues of \mathbf{A} , and a basis for each associated eigenspace. Is \mathbf{A} diagonalizable?
- (c) What is \mathbf{A}^m , where m is any integer?

3. Consider the system of linear equations:

$$\begin{aligned}x + y + z &= a \\2x + qy + z &= b \\x + 2y + 2z &= c\end{aligned}$$

- (a) Under what conditions (values of the constants q , a , b and c) will the system have (i) a unique solution; (ii) multiple solutions; (iii) no solution?
- (b) For case (ii), what is a basis for the associated homogenous system?
- (c) For case (i), what is the solution?

4. Given that A and B are two $n \times n$ matrices, answer the following:
- (a). Prove that AB and BA are non-singular if and only if A and B are both non-singular.
 - (b). If the rank of A is $l < n$ and the rank of B is $m < n$, what are the ranks of AB and BA .
 - (c). If the eigenvalues of AB are $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, show that BA has the same set of eigenvalues.
 - (d). Given an eigenvector of AB , find the eigenvector of BA corresponding to the same eigenvalue and vice versa.
 - (e). If AB and BA are diagonalizable, are they similar to each other? Prove or disprove.

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider any two real functions of a single real variable, $f(x)$ and $g(x)$, that are both continuous on the domain (a, b) . Prove the following statement:

$$\left(\int_a^b |f(x)g(x)| dx \right) \leq \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx$$

2. Consider the function of real variables:

$$f(x, y, z) = ax^2 + by^2 + cz^2$$

Find:

$$\int \int \int_R e^{f(x,y,z)^{3/2}} dx$$

where R is the region defined by:

$$f(x, y, z) \leq r^2$$

with $a, b, c,$ and r taken as arbitrary real constants greater than zero.

3. The expression $4x^2 + y^2 = 4$ defines an ellipse in \mathbb{R}^2 . Find the point on this ellipse that minimizes the distance to the line defined by: $y = 10 - x$.

4. Evaluate the following integral

$$\int_{-\infty}^{\infty} e^{-\alpha^2 x^2 + \beta x} dx.$$

where α and β are two constants.