DOCTORAL QUALIFYING EXAMINATION

SUMMER 2005

Advanced Calculus & Linear Algebra

NAME : ____

ID # : ____

There are four questions from Linear Algebra and four question from Advanced Calculus. For full credit, answer any **THREE questions from Linear Algebra** and any **THREE questions from Advanced Calculus**.

Start your answer on each question sheet. Attach all extra sheets you use to the appropriate sheet. Hand in all question sheets.

Date:	June 13, 2005
Time of Examination:	09:00 – 11:00
Place of Examination:	Physics P113

I.D.# ____________________

1. Consider the following four vectors in R^5:

$$u_1 = (1, -2, 1, 3, -1), \qquad u_2 = (-2, 4, -2, -6, 2),$$

$$u_3 = (1, -3, 1, 2, 1), \qquad u_4 = (3, -7, 3, 8, -1).$$

Find a *subset* of the vectors u_1, u_2, u_3, and u_4 which gives a basis for the subspace $W = \text{span}(u_1, u_2, u_3, u_4)$ of R^5.

I.D.# ______________________

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that a real matrix B commutes with A (that is, $AB = BA$) **if and only if** B has the form $sI + tA$, where s and t are real numbers and I is the 2×2 identity matrix.

I.D.# ____________________

3. Let A be a real non-singular matrix.

   (a) Prove that $A^T A$ is symmetric.

   (b) Prove that $A^T A$ is positive definite.

I.D.# ______________________

4. Prove that the matrix

$$\begin{pmatrix} 1 & 1.00001 & 1 \\ 1.00001 & 1 & 1.00001 \\ 1 & 1.00001 & 1 \end{pmatrix}$$

has one positive eigenvalue and one negative eigenvalue. *(Hint: $\lambda = 0$ is the other eigenvalue of the matrix.)*

I.D.# ____________________

5. Consider the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$, where $a_0, a_1, \ldots a_n$ are integers with $a_0 \neq 0$ and $a_n \neq 0$. Show that if the equation is to have a rational root p/q, then p must divide a_0 and q must divide a_n exactly.

I.D.# ______________________

6. Let a be a positive constant ($a > 0$). Prove that the function $f(x) = 1/x^2$ is *uniformly continuous* in the interval (a, ∞).

I.D.# ____________________

7. Let $a_1, a_2, a_3, \ldots$ be positive numbers.

(a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges.

(b) Prove that the converse of the statement in part (a) is false.

I.D.# ____________________

8. Show that

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx.$$