

**APPLIED MATHEMATICS and STATISTICS  
DOCTORAL QUALIFYING EXAMINATION  
in COMPUTATIONAL APPLIED MATHEMATICS**

**Spring 2009 (January)**

**(CLOSED BOOK EXAM)**

**This is a two part exam.**

**In part A, solve 4 out of 5 problems for full credit.**

**In part B, solve 4 out of 5 problems for full credit.**

Indicate below which problems you have attempted by circling the appropriate numbers:

<b>Part A:</b>	1	2	3	4	5
<b>Part B:</b>	6	7	8	9	10

**NAME** \_\_\_\_\_

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 28th, 2009

Time: 9:00 – 1:00 PM

Place: TBA

**A1.** The altitude of a mountain is given by

$$z = e^{-(x^4+4y^2)}.$$

Describe paths of rain water streams flowing down this mountain assuming that these paths are orthogonal to the contour lines (level curves) of the mountain.

**A2.** As  $x \rightarrow 0+$ , find the leading behavior (up to two terms) of solutions of the differential equation

$$x^6 y'' = e^x y.$$

**A3.** Solve the initial/boundary value problem by Fourier series.

$$u_{tt} - u_{xx} = 0, \quad 0 < x < \pi, t > 0.$$

$$u(x, 0) = 0, \quad 0 < x < \pi,$$

$$u_t(x, 0) = 1, \quad 0 < x < \pi,$$

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0.$$

**A4.**  $f(z)$  is analytic in  $N_R^*(z_0)$ ,  $z_0 \in C$ ,  $R > 0$ . Prove that the followings are equivalent:

(1)  $z_0$  is a removable singularity.

(2)  $f(z)$  is bounded in  $N_\delta^*(z_0)$ , ( $\delta \leq R$ ).

(3) The principle part of the Laurent expansion of  $f(z)$  has no terms.

**A5.**  $f(z)$  is analytic in a domain  $D$ ,  $z_0 \in D$ . Prove that If  $f'(z_0) \neq 0$ , there exists a neighborhood  $N_\rho(z_0) \subset D$ ,  $f(z)$  is univalent in  $N_\rho(z_0)$ . (Note: A function  $f(z)$  is said to be *univalent* in a domain  $D$  if it is analytic in  $D$  and assumes no value more than once in  $D$ ; that is,  $f(\xi) = f(\eta)$ ,  $\xi, \eta \in D$ , implies  $\xi = \eta$ .)

**B6.**

Let  $A \in \mathbb{C}^{m \times m}$  be a Hermitian matrix and  $\mathbf{q} \in \mathbb{C}^m$  be a vector with  $\|\mathbf{q}\|_2 = 1$ .

- a) Show that there exists a unitary matrix  $Q \in \mathbb{C}^{m \times m}$  whose **first** column is  $\mathbf{q}$ , such that  $Q^* A Q$  is a tridiagonal matrix.
- b) Show that there exists a unitary matrix  $\tilde{Q} \in \mathbb{C}^{m \times m}$  whose **last** column is  $\mathbf{q}$ , such that  $\tilde{Q}^* A \tilde{Q}$  is a tridiagonal matrix.

**B7.** Let  $\mathbf{A}$  be an  $m \times n$  real matrix. Consider the symmetric eigenvalue problem

$$\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}.$$

- a) Show that if  $\lambda$ ,  $\mathbf{u}$ , and  $\mathbf{v}$  satisfy this relationship, with  $\mathbf{u}$  and  $\mathbf{v}$  suitably normalized, then  $|\lambda|$  is a singular value of  $\mathbf{A}$  with corresponding left and right singular vectors  $\mathbf{u}$  and  $\mathbf{v}$ , respectively.
- b) Is solving this eigenvalue problem a good way to compute the SVD of the matrix  $\mathbf{A}$ ? Why?

**B8.** Let  $f \in C^1(\mathbb{R})$ .

a) Find the Newton form of the lowest order polynomial,  $P(x)$ , that satisfies

$$P(x_0) = f_0, \quad P'(x_0) = f'_0, \quad P(x_1) = f_1, \quad P'(x_1) = f'_1,$$

where  $f_i \equiv f(x_i)$  and  $f'_i \equiv f'(x_i)$ ,  $i = 0, 1$ , and  $x_0 < x_1$

b) What is the lowest order of the polynomial,  $Q(x)$ , that satisfies

$$Q(x_0) = f_0, \quad Q'(x_0) = f'_0, \quad Q'(x_1) = f'_1, \quad Q(x_2) = f_2,$$

with  $x_0 < x_1 < x_2$ ?

c) Using Lagrange polynomials, find the form of the lowest order polynomial,  $R(x)$ , that satisfies

$$R'(x_0) = f'_0, \quad R'(x_1) = f'_1, \quad R'(x_2) = f'_2, \quad R'(x_3) = f'_3,$$

with  $x_i \neq x_j$  when  $i \neq j$ ?

**B9.** Consider the scalar product

$$(f, g) \equiv \int_{-1}^1 f(x)g(x)dx$$

for functions  $f, g \in C[-1, 1]$ .

- a) When  $f(x)$  and  $g(x)$  are polynomials in  $\Pi_n$ , show that the scalar product can be numerically evaluated exactly (ignoring finite-precision errors) via the expression

$$(f, g) \equiv \sum_{i=0}^n \gamma_i f(\xi_i)g(\xi_i)$$

- b) Identify the points  $\xi_i$ .
- c) Express the coefficients  $\gamma_i$  in terms of Lagrange polynomials.

**B10.** Analyze the consistency, root condition and convergence of the linear multistep method

$$u_n = u_{n-4} + \frac{4h}{3} [2f_{n-1} - f_{n-2} + 2f_{n-3}]$$

for solving the ODE  $y'(x) = f(x, y(x))$ .