APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2010 (January)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A: 1 2 3 4 5
Part B: 6 7 8 9 10

NAME _________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate
question number at the top of any extra pages used to answer any question. Hand in all answer
pages.

Date of Exam: January 27th, 2010
Time: 9:00 – 1:00 PM
Place: Physics P113
A1. Classify the points $x = 0$ and $x = \infty$ of the following differential equations:

(a) $x^3 y''' = y$
(b) $y''' = x^3 y$
(c) $x^2 y'' = e^{1/x} y$
(d) $(\tan x) y' = y$
(e) $y'' = (\ln x) y$
A2. Find the general solution to the following system of ODE’s

\[
\begin{align*}
\frac{dx}{dt} &= -2x - 9y \\
\frac{dy}{dt} &= x + 4y \\
\frac{dz}{dt} &= x + 3y + z
\end{align*}
\]
A3. Solve the Cauchy problem

\[ yu_x + u_y - u^2 = 0 \]

with \( u(x,0) = h(x) \).
A4. \( f(z) = |z| \sin z = |z|(\sin x \cosh y + i \cos x \sinh y) \), \( g(z) = |z| \).

(a) Where \( g(z) \) is differentiable?

(b) Find all the roots of \( f(z) = 0 \).

(c) Show that: \( f(z) \) is not differentiable at \( z_0 \) if \( z_0 \) is not a root of \( f(z) = 0 \).

(d) Show that: \( f(z) \) is differentiable at \( z_0 \) if \( z_0 \) is a root of \( f(z) = 0 \). Evaluate \( f'(z_0) \).
A5. Given a simple closed contour $C$. $D$ is the exterior domain of $C$, $f(z)$ is analytic in $D$ and continuous in $D$, $\lim_{z \to \infty} z^2 f(z) = a$, where $a$ is a constant. Prove that

$$\int_C f(z) \, dz = 0.$$
B6. Let $A \in \mathbb{C}^{m \times m}$ be a Hermitian positive definite matrix with Cholesky factorization $A = R^* R$.

(a) Show that $\sqrt{\|A\|_2} = \|R\|_2$.

(b) Show that $\sqrt{\kappa_2(A)} = \kappa_2(R)$, where $\kappa_2$ denotes the condition number measured in 2-norm.
B7. Let $A \in \mathbb{C}^{m \times m}$ be skew Hermitian, i.e., $A^* = -A$
(a) Show that the eigenvalues of $A$ are purely imaginary.
(b) Argue that $A$ has a set of orthonormal eigenvectors.
(c) Show that $I - A$ is nonsingular.
B8. Newton’s method for solving a scalar nonlinear equation $f(x) = 0$ requires computation of the derivative of $f$ at each iteration. Suppose that we instead replace the true derivative with a constant $d$, that is, we use the iteration scheme

$$x_{k+1} = x_k - f(x_k)/d.$$ 

a) Under what condition on the value of $d$ will this scheme be locally convergent?

b) What will be the convergence rate, in general?

c) Is there any value for $d$ that would still yield quadratic convergence?
B9. Derive an open two-point Newton-Cotes quadrature rule for the interval \([a, b]\).

a) What are the resulting nodes and weights?

b) What is the degree of the resulting rule?
Consider the boundary value problem

$$-u'' + u = \sin t, \quad 0 < t < 1$$

with boundary conditions

$$u(0) = u(1) = 0.$$

a) Suppose you will be solving it using a finite difference method with centered difference approximation to $u''$ with a mesh size $h = \frac{1}{4}$ (i.e., $n = 4$). Derive the linear system for it.

b) For very small $h$ (i.e., very large $n$), what numerical method would you use to solve the resulting linear system? Justify your answer in terms of efficiency and stability.