

DOCTORAL QUALIFYING EXAMINATION

SPRING 2002

Advanced Calculus & Linear Algebra

NAME: \_\_\_\_\_

ID#: \_\_\_\_\_

Start your answer on each question sheet. Attach all extra sheets you use to the appropriate sheet. Hand in all question sheets.

Date: JANUARY 23, 2002

Time of Exam: 1-3PM

Place of Exam: Physics Building, Room P-116

ID#: \_\_\_\_\_

1. Calculate the following integral:

$$\int_0^{\pi} \frac{x dx}{a - \sin x}, \quad a > 1.$$

ID#: \_\_\_\_\_

2. (a) Prove that  $(1-x)e^{-x} < 1$ , for  $0 < x < 1$ .

(b) Further, prove that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0.$$

(If you use Stirling's formula, you must derive it.)

ID#: \_\_\_\_\_

3. Let  $f(x)$  and  $g(x)$  be two real-valued functions of  $x$ , which are continuous for  $-1 \leq x \leq 2$ . Prove that

$$\left( \int_0^1 |f(x)g(x)| dx \right)^2 \leq \int_0^1 |f(x)|^2 dx \int_0^1 |g(x)|^2 dx.$$

ID#: \_\_\_\_\_

4. Let  $A$  be a  $n \times n$ -matrix of real numbers.

- (a) Prove that  $(A^T)^{-1} = (A^{-1})^T$ , if  $A$  is nonsingular. ( $T$  is the transpose.)
- (b) Suppose further that  $A$  is symmetric and positive definite, i.e.,  $x^T Ax > 0$  for every vector  $x \neq 0$ . Prove that  $P^{-1}AP$  is also positive definite.

ID#: \_\_\_\_\_

5. Calculate the lengths of the major and the minor axes of the ellipse

$$17x^2 - 12xy + 8y^2 = 4.$$

ID#: \_\_\_\_\_

6. For a vector  $u$  in  $R^n$ , define  $\|u\|_2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$ , where  $u = (u_1, u_2, \dots, u_n)$  and for a matrix  $A$ ,  $\|A\|_2 = \sup_{\|u\|_2 \leq 1} \|Au\|_2$ . Let  $A$  be a real symmetric  $n \times n$ -matrix. Prove that  $\|A\|_2$  is the absolute value of the eigenvalue of  $A$  with the largest magnitude.