DOCTORAL QUALIFYING EXAMINATION

SPRING 2005

Advanced Calculus & Linear Algebra

NAME : ______________________________

ID # : ______________________________

There are four questions from Linear Algebra and four question from Advanced Calculus. For full credit, answer any **THREE questions from Linear Algebra** and any **THREE questions from Advanced Calculus**.

Start your answer on each question sheet. Attach all extra sheets you use to the appropriate sheet. Hand in all question sheets.

Date:	January 24, 2005
Time of Examination:	09:00 - 11:00
Place of Examination:	SBS S228

I.D.# ____________________

1. (a) A square matrix A is **nilpotent** if $A^k = 0$ for some positive integer k. Show that if $n \times n$ matrices A and B are both nilpotent *and* $AB = BA$, then $A + B$ is also nilpotent. Point out where in your proof you use the fact that A and B commute (*i.e.*, that $AB = BA$).

   (b) Give an example of a 2×2 **nilpotent** matrix A and a 2×2 **nilpotent** matrix B such that $A + B$ is **not** nilpotent.

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2. Let U and W be subspaces of a vector space V. Prove that $U \cup W$ is a subspace of V **if and only if** $U \subseteq W$ or $W \subseteq U$.

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3. Let W be the subspace of R^4 spanned by the vectors

$$v_1 = (1,1,1,1), \quad v_2 = (1,2,3,2), \quad v_3 = (4,0,0,0).$$

If $u = (4,0,4,2)$, find $proj(u, W)$, the projection of u onto W. (In other words find $w \in W$ which minimizes $\|u - w\|$).

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4. Find a real 2×2 symmetric matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and with eigenvector $u_1 = (1, 2)$ belonging to $\lambda_1 = 2$.

I.D.# ____________________

5. Prove that $\sqrt[3]{2}$ is an irrational number.

I.D.# ____________________

6. Let the sequence $\{u_n\}_{n=1}^{\infty}$ be defined as follows: $u_1 = 1$ and $u_{n+1} = \sqrt{2u_n}$ for $n = 1, 2, 3, \cdots$.

(a) Prove that $\lim_{n\to\infty} u_n$ exists.

(b) Find the limit in (a).

I.D.# ______________

7. If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B \neq 0$, prove directly that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$. (Note, "prove directly" means give an "ϵ, δ" proof).

I.D.# ____________________

8. Test each of the following improper integrals for convergence. Indicate clearly what your conclusion is (*convergent or divergent*) and how you reach that conclusion.

   (a) $\int_0^\infty \frac{x+\sqrt{x}}{2x^3-1}dx.$

   (b) $\int_1^\infty \frac{3^x}{x4^x}dx.$

   (c) $\int_1^\infty \frac{1+e^{-x}}{x}dx.$

   (d) $\int_0^\pi \frac{\sin x}{2x^3}dx.$