

AMS Common Exam - Part A, May 2006

Name: _____ ID Num. _____

Part A: _____ / 75

Part B: _____ / 75

Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider the system of linear equations:

$$w + x + y + z = 3$$

$$w + x - y - z = 2$$

$$w - x + y - z = 1$$

$$w + x - y + z = 0$$

- (a) Find the dimension and a basis of the solution space to the associated homogenous system.
- (b) Find the general solution for the given non-homogenous system.

2. Given $F = x^3 - y^3$ and $G = (x - y)(x^2 + xy + y^2)$, $F - G = 0$ for all scalars $x, y \in \mathbb{R}$.
- (a) What is $F - G$ for $x = \mathbf{X}, y = \mathbf{Y} \in \mathbb{R}^{n \times n}$ (the set of $n \times n$ square matrices)?
 - (b) Define a subspace of $\mathbb{R}^{n \times n}$ such that $F - G = 0$ for all elements x, y of the subspace.

3. Consider the subspace of \mathbb{R}^4 defined by $V = \text{span}(S)$, where:

$$S = \left\{ \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 6 & k & 3 \\ 4 & 8 & 5 & 9 \end{pmatrix} \right\}$$

- (a) For what values of k will $\dim(V) = 3$? What other values can $\dim(V)$ take?
- (b) Given $\dim(V) = 3$, find an orthonormal basis for V .
- (c) Given $\dim(V) = 3$, is the vector $(1, 3, 2, 1)$ in V ?

4. Given the linear transformation $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

$$F(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$$

- (a) Find a matrix representation, \mathbf{A} , of F .
- (b) Find all eigenvalues of \mathbf{A} , and a basis for each associated eigenspace. Is \mathbf{A} diagonalizable?
- (c) What is \mathbf{A}^m , where m is any integer?

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider the implicit function $y = f(x)$ in \mathbb{R}^2 defined by:

$$x^2 - xy + 1 = 0$$

- (a) Describe the behavior of $y = f(x)$ in the neighborhood of $x = 0$, and in the limits of $x \rightarrow \pm\infty$.
- (b) Find all local maxima and minima of $f(x)$. Are any of these the global maximum or minimum?
- (c) Sketch the curve. What are the bounds, if any, on $f(x)$?

2. Consider the function defined by:

$$f(x) = \left\{ \begin{array}{ll} \cos(x); & \text{if } x < 0 \\ 1; & \text{if } x = 0 \\ e^{-x}; & \text{if } x > 0 \end{array} \right\}$$

- (a) Comment on the continuity and smoothness of $f(x)$.
- (b) What is the total area lying between the x -axis and $f(x)$ on the domain $(-\frac{3\pi}{2}, +\infty)$?
- (c) What is the volume of the solid formed by rotating this region around the x -axis (one full revolution)?

3. Consider the relations:

$$\begin{aligned}u &= \frac{x+y}{1-xy} \\v &= \tan^{-1}(x) + \tan^{-1}(y)\end{aligned}$$

- (a) Find the Jacobian determinant $\frac{\partial(u,v)}{\partial(x,y)}$.
- (b) Find the functional relationship between u and v , or explain why such a relationship does not exist.
- (c) Is the transformation $F : (x, y) \rightarrow (u, v)$ a valid coordinate transformation on \mathbb{R}^2 ? Explain.

4. Consider the three cylinders of equal width defined by:

$$\begin{aligned}x^2 + y^2 &= r^2 \\x^2 + z^2 &= r^2 \\y^2 + z^2 &= r^2\end{aligned}$$

- (a) What is the volume of intersection of any two of these cylinders?
- (b) What is the volume of intersection of all three?