

DO THREE OUT OF FOUR LINEAR ALGEBRA PROBLEMS

Linear Algebra

1. Consider the system of linear equations:

$$\begin{aligned}x + y + kz &= 0 \\x + ky + z &= a \\kx + y + z &= b\end{aligned}$$

- (a) For what values of the constants k , a , and c will the system have (i) a unique solution; (ii) multiple solutions; (iii) no solution?
- (b) When there is a unique solution, what is it?
- (c) When there are multiple solutions, find the dimension and a basis of the solution space to the associated homogenous system.

2. Consider the upper triangular matrix \mathbf{A} , defined by:

$$\mathbf{A} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

- (a) Find an expression for \mathbf{A}^n , where n is any positive integer.
- (b) Find an expression for \mathbf{A}^n , where n is any *negative* integer.
- (c) Comment on the meaning of \mathbf{A}^0 .

3. Consider the subspace of \mathbb{R}^4 defined by $V = \text{span}(S)$, where:

$$S = \left\{ \begin{array}{l} (3, 0, 3, 0), \\ (4, 1, 4, 1), \\ (1, -1, 1, 1), \\ (2, 2, 2, 4) \end{array} \right\}$$

- (a) What is the dimension of V ?
- (b) Find an orthonormal basis for V .
- (c) Is the vector $(1, 2, 3, 4)$ in V ?

4. Given the linear transformation, F , on \mathbb{R}^3 defined by:

$$F(x, y, z) = (3x - y, -x + 2y - z, -y + 3z)$$

- (a) Find all eigenvalues of the transformation, and a basis for each associated eigenspace. Is F diagonalizable?
- (b) Find a transformation G , such that $G \circ G = F$.

Advanced Calculus

1. Consider the curve in \mathbb{R}^2 defined by:

$$f(x) = \frac{x^2 + 2x - 4}{x^2}$$

- (a) Find any points of discontinuity and the limits of $f(x)$ in the neighborhood of these points, as well as the limits as $x \rightarrow \pm\infty$.
- (b) Find the points $(x, f(x))$ corresponding to all roots, maxima, minima and inflection points of $f(x)$.
- (c) Sketch the curve. What are the bounds, if any, on $f(x)$?

2. Consider the function $f(x) = x^2 \sin(x - \pi)$.

- (a) What is the total area lying between the x-axis and $f(x)$ on the domain $(-\pi, +\pi)$?
- (b) What is the volume of the solid formed by rotating this region around the y-axis (one full revolution)?

3. The plane and cylinder in \mathbb{R}^3 defined by:

$$x - y + z = 1$$

and:

$$x^2 + y^2 = 1$$

intersect in a space curve. Find the maximum and minimum values of the function:

$$f(x, y, z) = x + 2y + 3z$$

on this curve.

4. Consider the function:

$$f(x, y, z) = x^2 + 4y^2 + 9z^2$$

Using an appropriate coordinate substitution, find the value of:

$$\iiint_R e^{(f(x,y,z))^{\frac{1}{2}}} dV$$

where R is defined by $f(x, y, z) \leq 16$.