DO THREE OUT OF FOUR PROBLEMS FROM AMS 504 AND 505

504 Question 1 – AMS QUALIFYING EXAM – January, 2006

Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. Compute

$$\lim_{n\to\infty} (a_1^n + a_2^n + \cdots + a_k^n)^{1/n}.$$

504 Question 2 AMS QUALIFYING EXAM January, 2006

For $n = 0, 1, 2, ...,$ and $x \in \mathbb{R}$, define $P_n(x)$ by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Prove that P_n has n distinct zeros in $(-1, 1)$.

AMS 505

1. Let A be a $n \times n$ matrix with elements $a_{ij}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$, which are real numbers with absolute value ≤ 1. Moreover, for each $j, j = 1, 2, \cdots, n$,

$$\sum_{k=1}^{n} | a_{jk} |^2 \leq 1.$$

Show that $| \; det(A) \; | \leq 1$ and the equality holds only when the rows of the matrix are orthogonal pairwise.

2. A monic polynomial $m(x)$ is called a minimal polynomial for a linear operator T on a finite-dimensional linear vector space V if the operator $m(T)$, defined in the customary way , $T^0 = I, T^{k+1} = T^k$ annihilates every vector in V, i.e. the null space of $m(T)$ is the entire space V.

   (a) Prove that $m(x) = x - 1$ if and only if $T = I$, where I is the Identity Operator.

   (b) Prove that unless a $n \times n$ matrix A is an identity matrix, if it has an eigenvalue 1 with algebraic multiplicity n, it must be defective (can not have the geometric multiplicity n).