AMS Common Exam - Part A, May 2007

Name: _____ ID Num. _____

Part A: _____ / 75

Part B: _____ / 75 Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Given the linear transformation, $F : \mathbb{R}^4 \to \mathbb{R}^3$ defined by:

$$F(w, x, y, z) = (w + 3x + 6y + z, w + x - y - 2z, 3w + 5x + 4y - 3z)$$

   (a) Find a matrix representation of F.

   (b) Find the dimension of, and a basis for, both the kernel and the image of F.

2. Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Find the eigenvalues of $\mathbf{A}$.

(b) Find matrices, $\mathbf{P}$ and $\mathbf{D}$, such that $\mathbf{A} = \mathbf{PDP}^T$, where $\mathbf{D}$ is diagonal.

3. Consider the subspace of $\mathbb{R}^4$ spanned by the set S:

$$S = \{ \begin{matrix} (2, & 2, & 2, & 4) & , \\ (2, & 1, & 2, & 5) & , \\ (1, & 2, & 1, & 1) & , \\ (3, & 1, & 1, & 4) & \} \end{matrix}$$

(a) Find the dimension and a basis for $V = \text{span}(S)$.

(b) Is the vector $\vec{w} = (3, 5, 1, 0)$ in V? If so, determine the coordinates of $\vec{w}$ relative to the basis found in (a).

4. Consider any non-singular matrix $\mathbf{A}$, and any similar matrix $\mathbf{B}$.

   (a) Prove that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$.

   (b) Prove that $\mathbf{B}$ is also non-singular, and has the same eigenvalues as $\mathbf{A}$.

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider the polynomial:

$$p(x) = x^5 - 2x^4 - x - 3$$

   (a) Show that $p(x)$ has a root in the interval $(2, 3)$.

   (b) Use Newton's Method to find this root to within an error of 0.1 (in x).

2. Consider the function of $\mathbb{R}^2$ defined by:

$$f(x, y) = x^2 + 12xy + 2y^2$$

(a) Is $f(x, y)$ bounded above, below, or both? What are the bounds?

(b) Find the absolute maximum and mininum of $f(x, y)$ on the domain:

$$\{(x, y) : 4x^2 + y^2 <= 25\}$$

3. Consider the function $f(x) = sin(x^3)e^{-x^2}$.

   (a) Calculate $f'(x)$ and $f''(x)$.

   (b) Evaluate $f(x)$, $f'(x)$, and $f''(x)$ at x=0, and comment on the behavior of the original function, $f(x)$, at this point.

   (c) *Estimate* the location of a relative maximum and a relative minimum. Compute $f(x)$, $f'(x)$, and $f''(x)$ at these points and comment.

4. Consider the two cylinders of equal width defined by:

$$x^2 + y^2 = r^2$$
$$y^2 + z^2 = r^2$$

(a) What is the volume of intersection of these cylinders?