

# AMS Common Exam - Part A, January 2007

Name: \_\_\_\_\_

ID Num. \_\_\_\_\_

Part A: \_\_\_\_\_ / 75

Part B: \_\_\_\_\_ / 75

Total: \_\_\_\_\_ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

**Good Luck!**

## Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider the  $2 \times 2$  matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

- (a) Find a general expression for  $\mathbf{A}^m$ ,  $\forall m \in \mathbb{N}$  ( $m$  is any natural number).
- (b) Extend this result to  $\mathbf{A}^q$ ,  $\forall q \in \mathbb{Z}$  ( $q$  is any integer). For what values of  $a$  and  $b$  is this result valid?

2. Consider the linear transformation,  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by:

$$F(x, y, z) = (2x - 3y - 2z, -3x + 3y - 3z, -2x - 3x + 2z)$$

- (a) Find the matrix representation of  $F$ , relative to the usual basis of  $\mathbb{R}^3$ .
- (b) If possible, find a diagonal matrix that is also a representation of  $F$ .
- (c) What is the basis for which this diagonal matrix is the representation of  $F$ .

3. Let  $\mathbf{A}$  and  $\mathbf{B}$  be any two real non-singular matrices.
- (a) Show that  $\mathbf{AB}$  and  $\mathbf{BA}$  are similar to one another, and thus represent the same linear transformation relative to two different bases.
  - (b) What are the change-of-basis matrices between these two bases?
  - (c) Show that, if diagonalizable,  $\mathbf{AB}$  and  $\mathbf{BA}$  must have the same eigenvalues.

4. Consider the following system of linear equations:

$$\begin{aligned}2w + 4x - 5y + 3z &= 2 \\ -3w - 6x + 7y - 4z &= -1 \\ 6w + 12x - 13y + 7z &= -2 \\ 5w + 10x - 11y + 6z &= -1\end{aligned}$$

- (a) Find the general solution to this system, if one exists.
- (b) Find the dimension and a basis for the associated homogenous system.
- (c) Comment on the relationship between these two solutions.

## Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider the function of two variables,  $f(x, y) = 2y(1 - x)$ .
  - (a) Find all critical points of  $f(x, y)$  and classify them as maxima, minima, or saddle points. What are the global maximum and minimum of  $f(x, y)$ ?
  - (b) What are the maximum and minimum values of  $f(x, y)$  on the curve defined by,  $x^2 + \left(\frac{y}{2}\right)^2 - 1 = 0$ .
  - (c) Make a sketch of the  $xy$ -plane including:
    - the curve defined in (b);
    - the locations of all the points found in (a) and (b);
    - the level curves of  $f(x, y) = c$ , where  $c$  is the value of  $f(x, y)$  at each point found in (a) and (b).

2. Consider the following expression:

$$dw = [(1 - y) \cos(x) + xy \sin(x)] dx + [4yz e^{y^2 z} - x \cos(x)] dy + \left[ 2y^2 e^{y^2 z} + \frac{2}{2z + 3} \right] dz$$

- (a) Demonstrate whether or not this is an exact differential.
- (b) Find a function  $w = F(x, y, z)$ , such that  $dw$  is as given, or explain why this is not possible.

3. Consider the function of three variables:

$$f(x, y, z) = \frac{1}{(z+1)^2} \sin\left(\frac{\sqrt{x^2 + y^2}}{2}\right)$$

- (a) Characterize the zero-level surfaces of  $f(x, y, z)$  (*i.e.* the surfaces for which  $f(x, y, z) = 0$ ). Include a sketch.
- (b) What is the value of the integral of  $f(x, y, z)$  above the  $xy$ -plane and bounded by the two zero-level surfaces nearest (but excluding) the origin.

4. Using mathematical induction, prove:

$$2^n = \sum_{i=0}^n {}_n C_i$$

for any natural number,  $n$ , where  ${}_n C_k = \frac{n!}{k!(n-k)!}$  is the binomial coefficient (note that  $0! = 1$ ).