

AMS Common Exam - Part A, June 2008

Name: _____

ID Num. _____

Part A: _____ / 75

Part B: _____ / 75

Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider the product of any two n -square matrices:

$$\mathbf{C} = \mathbf{AB}$$

- (a) Prove that \mathbf{C} is non-singular if and only if both \mathbf{A} and \mathbf{B} are non-singular.
- (b) Prove that the rank of \mathbf{C} must be less than or equal to the smaller of the ranks of \mathbf{A} and \mathbf{B} . In other words:

$$\text{rank}(\mathbf{C}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B}))$$

2. Consider the system of linear equations:

$$\begin{array}{rccccr} x & & & + & pz & = & a \\ x & + & qy & + & z & = & a \\ rx & & & + & z & = & a \end{array}$$

- (a) For what values of $a, p, q, r \in \mathbb{R}$ will the system have (i) a unique solution, (ii) no solution, (iii) infinite solutions.
- (b) In the case of a unique solution, find a general expression for this solution.

3. Consider the matrix:

$$\mathbf{M} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- (a) Find all eigenvalues of \mathbf{M} , and a basis for each associated eigenspace.
- (b) Find a general expression for \mathbf{M}^n , for any natural number n .

4. Consider the linear mapping $F : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ represented by the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

relative to the usual bases of \mathbb{R}^4 and \mathbb{R}^3 .

(a) What is the rank and nullity of F ?

(b) Let V be the subspace of \mathbb{R}^4 spanned by:

$$\{(1, 1, 1, 1), (1, 1, 2, 2), (1, 2, 3, 4), (2, 3, 5, 6)\}$$

Find the dimension and a basis for the image of V under F .

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider an arbitrary polynomial:

$$P(x) = \sum_{i=0}^n a_i x^i$$

where $a_i \in \mathbb{R}$ and $n \in \mathbb{Z}$ ($n > 0$).

- (a) Prove that $P(x)$ is continuous on the domain $(-\infty, \infty)$.
- (b) Prove that any derivative or antiderivative of $P(x)$ is also continuous on $(-\infty, \infty)$.

2. Consider the function of a single real variable:

$$f(x) = \sin\left(\frac{1}{x}\right)$$

- (a) Identify any points of discontinuity of $f(x)$ on the domain $(-\infty, \infty)$. What is the limit of $f(x)$ at these points, and at $\pm\infty$?
- (b) Find $f'(x)$ and $f''(x)$, and write a general expression for all maxima/minima of $f(x)$.
- (c) Sketch $f(x)$, labeling the coordinates of one maximum and one minimum.

3. Consider the function of problem 2:

$$f(x) = \sin\left(\frac{1}{x}\right)$$

- (a) Prove that the improper integral, $\int_0^a f(x)dx$, is convergent (where $a > 0$).
- (b) Would you expect the integral $\int_b^\infty f(x)dx$ ($b > 0$) to be convergent? Explain why or why not.
- (c) Comment on the value of $\int_{-a}^a f(x)dx$. What about $\int_{-\infty}^\infty f(x)dx$?

4. Consider the function of three real variables:

$$\rho(x, y, z) = \frac{c^2}{z^2(a^2x^2 + b^2y^2)^{\frac{3}{2}}}$$

where a , b , and c are arbitrary real constants.

- (a) Identify any points of discontinuity on $\rho(x, y, z)$; what is the limit of the function in the neighborhood of these points?
- (b) What is the value of the definite integral of $\rho(x, y, z)$ over the domain:

$$a^2x^2 + b^2y^2 > \epsilon^2 \quad z^2 > \zeta^2$$

- (c) What is the behavior of the integral as ϵ and ζ approach zero?