AMS Common Exam - Part A, January 2008

Name: ______________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Consider a parallelogram in $\mathbb{R}^3$ with edges defined by the vectors $\vec{x}$ and $\vec{y}$, and a parallelepiped formed by vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$.

   (a) Find an expression for the area of the parallelogram in terms of inner vector products.

   (b) Find an expression for the volume of the parallelepiped in similar terms.

2. Consider the set of functions of the form $ae^x + be^{-x} + c$, that is:

$$\mathbb{T} = \left\{ t(x) | t(x) = ae^x + be^{-x} + c, \forall a, b, c \in \mathbb{R} \right\}$$

(a) Show that $\mathbb{T}$ is a vector space.

(b) What is the dimension of and a basis for $\mathbb{T}$?

(c) What is the dimension of and a basis for the subspace of $\mathbb{T}$ spanned by:

$$\left\{ f(x) = e^x + 2e^{-x} + 1, g(x) = e^x - e^{-x} + 1, h(x) = e^x + 1 \right\}$$

3. Symmetry and anti-symmetry.

   (a) Show that a matrix $\mathbf{B}$ is symmetric if and only if there exists a matrix $\mathbf{A}$ such that $\mathbf{B} = \mathbf{A} + \mathbf{A}^T$.

   (b) Show that a matrix $\mathbf{D}$ is anti-symmetric if and only if there exists a matrix $\mathbf{C}$ such that $\mathbf{D} = \mathbf{C} - \mathbf{C}^T$.

   (c) Show that $\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M}$ for all square matrices, $\mathbf{M}$.

4. Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find all eigenvalues of $\mathbf{A}$, and an orthonormal basis for each associated eigenspace.

(b) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, or explain why this is not possible.

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Consider the function of two real variables:

$$f(x,y) = e^{\frac{-(x^2+y^2)}{2\sigma^2}}$$

where $\sigma > 0$ is a constant; this is an unnormalized Gaussian in two dimensions.

(a) Show that this function has a single critical point at $(0,0)$.

(b) What is the value of the improper integral of $f(x,y)$ over all space (that is, over the domain $x \in (-\infty, +\infty)$, $y \in (-\infty, +\infty)$).

(c) Using the result of (b), suggest a modification to $f(x,y)$ that will give a function whose integral over all space is 1; this is a normalized Gaussian surface.

2. Consider the sphere in $\mathbb{R}^3$ defined by:

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$$

(a) Using the method of Lagrange multipliers, find the minimum and maximum distances from this surface to the origin.

(b) Comment on the geometric relationship of these points with respect to the sphere and the origin.

3. Consider the derivative of x raised to a power:

$$\frac{d}{dx}x^p = px^{p-1}$$

(a) Using the definition of the derivative, show this is true when p is any positive integer.

(b) Similarly, show this is true when p is any negative integer.

(c) Comment on the case of $p = 0$.

4. Consider the following rational function:

$$g(x) = \frac{2x^4 - 3x^2 + 5x^2 - 4x + 2}{x^3 - x^2 + x - 1}$$

(a) Identify any points of discontinuity on $g(x)$; what are the limits of the function in the neighborhood of these points?

(b) Evaluate the indefinite integral $\int g(x)dx$.

(c) Evaluate the definite integral $\int_{-1}^{1} g(x)dx$, or provide an explanation why this is not possible.