

Doctoral Qualifying Examination

Fall 1999

Advanced Calculus & Linear Algebra

Name: _____

ID#: _____

Start your answer on each question sheet. Attach all extra sheets you may have used to the appropriate sheet. Hand in all question sheets.

Time: 8:30 – 10:30 AM, September 1, 1999

Place: SBS N106

ID #: _____

1. Let $f(x, y)$ be a real-valued continuous function defined in a region R . Further, $f(x, y)$ has continuous partial derivatives of second order (including mixed derivatives f_{xy} and f_{yx}). Prove that

$$f_{xy} = f_{yx},$$

at every interior point of the region. What may happen at the boundary of the region R ?

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2. Analytically (as opposed to graphically) determine the point on the ellipse

$$4x^2 + y^2 = 4,$$

which is closest to the straight line $x + y = 10$.

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3. Let $\{s_n\}$ denote a sequence of real numbers which converges to a real number a . Define a new sequence σ_n as follows :

$$\sigma_1 = s_1, \quad \sigma_k = \frac{s_1 + s_2 + \cdots + s_{k-1} + s_k}{k}, \quad k = 2, 3, \dots$$

Prove that the sequence $\{\sigma_k\}$ also converges to a .

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4. Consider the matrix mapping $A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ where

$$A = \begin{pmatrix} 1 & 3 & 6 & 1 \\ 1 & 1 & -1 & -2 \\ 3 & 5 & 4 & -3 \end{pmatrix}.$$

Find a basis and the dimension of (a) the image of A and (b) the kernel of A .

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5. Calculate the lengths of the major and the minor axes of the ellipse $17x^2 + 12xy + 8y^2 = 4$.

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6. Let V be a finite-dimensional linear vector space and let U be a linear vector space (not necessarily of finite dimension). Let $T : V \rightarrow U$ be a linear mapping. Prove that

$$\dim(V) = \dim(\text{Ker } T) + \dim(\text{Im } T),$$

where $\text{Ker } T$ and $\text{Im } T$ are respectively the kernel and the image of T .