APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2000

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Thurs., Sept. 7, 2000

Time: 1:00 – 5:00 PM

Place: MATH, 1-122A

A1. Prove that there is at most one solution to the following two-point boundary value problem:

$$y'' = 2y(1 + (y')^2) \; , \qquad y(0) = a, \quad y(1) = b \; .$$

A2. Suppose that A is a matrix of real constants, with all the eigenvalues of A having negative real parts. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map which can be expressed as $Ax + g(x)$, where

$$\lim_{x \to \mathbf{0}} \frac{\| g(x) \|}{\| x \|} = 0 .$$

Prove that the differential equation

$$\frac{dx}{dt} = f(x)$$

is asymptotically stable at $\mathbf{0}$.

A3. For the partial differential equation

$$u_{tt} - c^2 u_{xx} + 2\alpha u_t + \gamma u = 0,$$

where $\gamma > 0$ and $0 < \alpha \leq \sqrt{\gamma}$:

(a) Find the dispersion relation (angular frequency ω as a function of wave number k), phase velocity, and group velocity.

(b) Show that solutions decay exponentially in time, and find the rate of decay.

A4. Suppose that $\Omega \subseteq \mathbf{R}^n$ is open, f is a smooth function defined on Ω, and $x^* \in \Omega$. If $\ell > 0$ is small enough that the cube C_ℓ centered at x^* with sides of length ℓ is contained in Ω, let $\langle f \rangle_\ell$ denote the average value of f over C_ℓ.

(a) Show that

$$\Delta f(x^*) = -(24/\ell^2)\,[f(x^*) - \langle f \rangle_\ell] + \mathcal{O}(\ell^2)$$

as $\ell \to 0^+$. (Here $\Delta = \text{div grad}$.)

(b) Suppose that u satisfies the heat equation

$$u_t = \Delta u$$

on Ω for $t > 0$. Using the result of part (a), show that if the maximum of $u(\cdot, t)$ occurs at $x^* \in \Omega$, then $u_t(x^*, t) \leq 0$.

A5. Let $f(z) = 1/z$ for $0 < |z| < \infty$, and let $f(\infty) = 0$.

(a) Prove that $f(z)$ is analytic at $z = \infty$.

(b) Find the residue of $f(z)$ at $z = \infty$ and prove your claim.

B6. Consider the solution of the linear system

$$\begin{aligned} 5x_1 - 2x_3 &= -1 \\ -4x_1 + 8x_2 + 2x_3 &= 18 \\ 5x_2 + 9x_3 &= 37 \end{aligned}$$

by the Jacobi method. Give an estimate on the number of iterations needed to ensure that

$$\| x_\nu - x \|_\infty \leq 10^{-3}$$

if the iteration is started with $x_0 = (0, 0, 0)^T$.

B7. Show that Newton's method for the function

$$f(x) = x^n - a\,, \qquad x > 0\,,$$

where $n > 1$ and $a > 0$, converges globally to $a^{1/n}$.

B8. A second order ordinary differential equation

$$\frac{d^2y}{dx^2} + 101\frac{dy}{dx} + 100y = 0, \qquad y(0) = 1, \quad y'(0) = 2,$$

is to be solved through Euler's method. What is the maximum step size under which the numerical solution is stable? (Document the calculation leading to your answer.)

B9. Find the values of A_1, A_2, x_1, and x_2 such that the evaluation of the integral

$$I = \int_a^b f(x)dx$$

through

$$I \approx A_1 f(x_1) + A_2 f(x_2)$$

is exact when $f(x)$ is any polynomial of order 3 or less.

B10. A function $f(x)$ can be interpolated through Newton's forward divided difference

$$f(x) \approx f[x_0] + f[x_0, x_1](x - x_0) + \cdots f[x_0, x_1, \cdots, x_n](x - x_0) \cdots (x - x_{n-1}) \ .$$

Prove that the truncation error in this interpolation is

$$e(x) = f[x_0, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n) \ .$$

Show that the interpolation degenerates into Taylor's expansion if $x_0 = x_1 = \cdots = x_n$. Estimate the truncation error in the Taylor expansion.