APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2001

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., Aug. 29, 2001

Time: 12:00 – 4:00 PM

Place: MATH, 1-122A

A1. Let $A(t)$ be a $n \times n$ matrix of functions continuous on an interval $[a, b]$. Let $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ be n solutions of the system of differential equations

$$\frac{dx}{dt} = A(t)\, x(t) \ .$$

Prove that the Wronskian $W(x_1, x_2, \ldots, x_n)$ either vanishes identically or is never zero on the interval $[a, b]$.

A2. Consider the planar system of differential equations

$$\dot{x} = -y + (x - y)(x^2 + y^2 - 1) \ ,$$
$$\dot{y} = x + (x + y)(x^2 + y^2 - 1) \ .$$

a) Show that the origin $(0, 0)$ is the only equilibrium point.

b) Show that the unit circle $x^2 + y^2 = 1$ is a trajectory of this system.

c) Show that the trajectories with an initial point outside the unit circle move away from the origin as t increases.

d) What happens to the trajectories with an initial point inside the unit circle?

A3. Consider a bounded domain Ω in the plane with smooth boundary Γ. Given a function ϕ on Γ, one can solve the Dirichlet problem

$$\begin{aligned} \Delta f &= 0 \qquad \text{in } \Omega \, , \\ f &= \phi \qquad \text{on } \Gamma \end{aligned}$$

(where $\Delta f = \nabla \cdot \nabla f$). One can then evaluate $\partial_n f$, the normal derivative of f along Γ, which is a function on Γ. Given ϕ, $\partial_n f$ is determined, and we denote it by $N(\phi)$.

(a) Show that the mapping $\phi \mapsto N(\phi)$ is a linear transformation.

(b) Show that N is also symmetric:

$$\langle N(\phi_1), \phi_2 \rangle = \langle \phi_1, N(\phi_2) \rangle \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined by

$$\langle \phi_1, \phi_2 \rangle = \int_\Gamma \phi_1(s)\phi_2(s)\, ds \ .$$

A4. Let u be a C^2 solution of the parabolic equation

$$u_t = a(x,t)u_{xx} + 2b(x,t)u_x + c(x,t)u$$

in the rectangle $\Omega := \{ (x,t) : 0 < x < \ell \,, 0 < t < T \}$. Here a, b, and c are continuous, and $a(x,t) > 0$ for all $(x,t) \in \Omega$. Let $\partial'\Omega$ denote the "parabolic boundary" of Ω, *i.e.*, the set $\partial\Omega \backslash \{ (x,T) : 0 < x < \ell \}$.

(a) Assuming that $c(x,t) < 0$ for all $(x,t) \in \overline{\Omega}$, prove that

$$|u(x,t)| \le \sup_{\partial'\Omega} |u| \qquad \text{for all } (x,t) \in \overline{\Omega}.$$

(b) Show more generally that

$$|u(x,t)| \le e^{CT} \sup_{\partial'\Omega} |u| \qquad \text{for all } (x,t) \in \overline{\Omega},$$

where

$$C := \max\{ 0, \sup_{\overline{\Omega}} c \} \,.$$

A5. If $f(z)$ is analytic in $|z| \leq R$, $|a| < R$, and $|b| < R$, prove that

$$f(a) - f(b) = \frac{a-b}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz \ .$$

Use this result to give a proof of Liouville's theorem.

B6. Consider the matrix

$$A = \begin{bmatrix} 1 & 10^{-3} & 10^{-4} \\ 10^{-3} & 2 & 10^{-3} \\ 10^{-4} & 10^{-3} & 3 \end{bmatrix} .$$

a) Estimate its eigenvalues using the Bauer-Fike Theorem.

b) Estimate its eigenvalues using the Gershgorin Theorem.

c) Improve the estimate using Wilkinson's correction prodedure.

d) Estimate the condition number of A in the l_2-norm.

B7. The mixed hybrid finite element formulation using lowest order Raviart-Thomas elements yields the block linear system

$$\begin{bmatrix} A & -D^t & -B^t \\ D & 0 & 0 \\ B & 0 & I_s \end{bmatrix} \begin{bmatrix} Q \\ P \\ \Pi \end{bmatrix} = \begin{bmatrix} F_Q \\ 0 \\ F_\Pi \end{bmatrix}, \qquad (MH)$$

where: ^{t} denotes matrix transpose; A, D, B, and I_s are matrices; and Q, P, Π, F_Q and F_Π are column vectors. System (MH) is most efficiently solved by performing a block LU decomposition

$$LU \begin{bmatrix} Q \\ P \\ \Pi \end{bmatrix} = \begin{bmatrix} F_Q \\ 0 \\ F_\Pi \end{bmatrix},$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}, \qquad \text{and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{21} & U_{22} \\ 0 & 0 & U_{33} \end{bmatrix},$$

and solving

$$U \begin{bmatrix} Q \\ P \\ \Pi \end{bmatrix} = L^{-1} \begin{bmatrix} F_Q \\ 0 \\ F_\Pi \end{bmatrix}. \qquad (MH1)$$

a) Give the explicit forms for the entries in L and U in terms of the original matrices in (MH).

b) In (MH1), the solution subvector Π can be "back-solved for" first. Give the explicit form of the linear system which must be solved in this back-solve step for Π.

B8. From the definition of the Chebyshev polynomial $T_k(x)$ in $[-1,1]$:

$$T_k(\cos\theta) = \cos k\theta,$$

a) find the three-term recurrence relation among $T_{k+1}(x)$, $T_k(x)$ and $T_{k-1}(x)$.
Hint: Consider $\cos((k+1)\theta) + \cos((k-1)\theta)$.

b) Show that $T_i(x)$ and $T_j(x)$ $(i \neq j)$ are orthogonal with respect to the scalar product

$$< g, h >= \int_{-1}^{1} \frac{g(x)h(x)}{\sqrt{1-x^2}} dx \ .$$

c) Expand the function $f(x) = x^4$ in terms of Chebyshev polynomials.

B9. Given f_i and f_i' and points x_i, $i = 0, 1$.

a) Find an osculatory polynomial interpolation $p_3(x)$ such that $p_3(x_i) = f_i$, $p_3'(x_i) = f_i'$, $i = 0, 1$ and show that

$$p_3\left(\frac{x_0 + x_1}{2}\right) = \frac{1}{2}(f_0 + f_1) + \frac{x_1 - x_0}{8}(f_0' - f_1') \ .$$

b) Show that if we let

$$f\left(\frac{x_0 + x_1}{2}\right) = p_3\left(\frac{x_0 + x_1}{2}\right)$$

as the node condition for the middle point, $(x_0 + x_1)/2$, then Simpson's rule of integration gives

$$I_S = \frac{1}{2}(x_1 - x_0)(f_0 + f_1) + \frac{1}{12}(x_1 - x_0)^2(f_0' - f_1') \ .$$

B10. Design a second order accurate numerical algorithm to solve the following boundary value problem:

$$y'' = 0.5y - \frac{2(y')^2}{y}, \qquad y(0) = 2, \quad y(1) = 3 \ .$$

Explicitely show that your algorithm will calculate the numerical solution to second order accuracy in the step-size h.