APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2002

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Thurs., Sept. 5, 2002

Time: 11:00 – 3:00 PM

Place: SBS, N-118

A1. Consider the differential equation $dx/dt = 5(x-1)^{4/5}$.

a) Find the unique solution on the interval $[0, 1]$ with $x(0) = 0$.

b) For what values of α are there more than one solution on $[0, \alpha]$ satisfying $x(0) = -31$. Find at least two solutions for such α.

c) What is the reason that you have more than one solution in (b)?

A2. Consider a mass m suspended from a spring, of spring constant $k(> 0)$. Let x denote deflection from rest position. The resultant differential equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

a) Convert the differential equation into a system of first order differential equations by introducing the variable $v = x'$.

b) Discuss the stability of the equilibrium points of the system.

c) Sketch the phase portrait of the system in the x-v plane.

d) Let $B(x_\infty, v_\infty)$ be the collection of initial conditions whose solution converges to the point (x_∞, v_∞), i.e., $B(x_\infty, v_\infty) = \{(x(0), y(0)) \mid \lim_{t\to\infty}(x(t), v(t)) = (x_\infty, v_\infty)\}$. Find $B(0, 0)$ and $B(0, 1)$. Suppose the mass is put into motion with initial velocity $v(0) \neq 0$. What will happen to the mass after infinitely long time? Will the mass stop eventually? (Base your answer on the solution to the differential equation and not on physics.)

A3.

a) Prove that the fundamental solution for the Laplace equation is in $L^1(D)$ for any bounded domain $D \subset R^n$.

b) Derive a formal solution of the Dirichlet problem

$$\left.\begin{aligned} \triangle u = 0, \quad & x \in \Omega, \\ u(x) = f(x), \quad & x \in \partial\Omega, \end{aligned}\right\} \text{ where } \Omega = \{x = (x_1, x_2) \in R^2 : |x| < 1, \ x_2 > 0\},$$

and $f(x)$ is continuous in R^2 with $f(x_1, 0) = 0$.

A4. Let $u(x,t)$ be $C^2(R)$ in x and $C^1(0,\infty)$ in t. Assume u satisfies the heat equation $u_t = a^2 u_{xx}$.

a) Prove that for $l > 0$ and any $t > 0$,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^l u^2 dx\right) = a^2[uu_x]_0^l - a^2\int_0^l (u_x)^2 dx.$$

b) What type of initial-boundary value conditions (other than Dirichlet boundary conditions at $x = 0, l$) for the heat equation would guarantee a unique solution? Prove your proposal.

A5. Expand the analytic function

$$f(z) = e^{z^2} \int_0^z e^{-t^2}\, dt$$

in a power series of the form $\sum_{n=0}^{\infty} a_n z^n$, where a_n is constant, $n = 0, 1, 2, \ldots$.

B6.

a) Show that the following real matrix is a positive definite.

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \qquad -1 < \epsilon < 1.$$

(*Hint*: Use the definition of a positive definite matrix.)

b) Give estimates based on Gerschgorim's theorem for the eigenvalues of the above matrix.

b) Suppose a square matrix A is *strictly row diagonally dominant*, which means that for each k

$$|a_{kk}| > \sum_{j \neq k} |a_{kj}|.$$

Prove that A is non-singular. (*Hint*: Use Gerschgorin's theorem.)

B7. The equation $x^4 - \sin(\pi x/2) = 0$ is solved numerically by the fixed point iteration $x_{n+1} = g(x_n)$ for a suitable function $g(x)$ starting from $x_0 = 1.1$.

a) Will it succeed in finding the root $x = 1$ when $g(x) = \frac{\sin(\pi x/2)}{x^3}$? Why or why not?

b) Find the function $g(x)$ given by Newton's method.

B8. Let $g(x) \equiv f[x_0, \cdots, x_k, x]$.

a) Using induction, prove that

$$g[y_0, \cdots, y_n] = f[x_0, \cdots, x_k, y_0, \cdots, y_n]$$

b) Prove that

$$g^{(n)}(x) = n! f[x_0, \cdots, x_k, \overbrace{x, \cdots, x}^{n+1 \text{ times}}]$$

B9. When the scalar product of two functions $g(x)$ and $h(x)$ is given by

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)dx,$$

then the resulting orthogonal polynomials are the Legendre's polynomials. The first two polynomials are $P_0(x) = 1$ and $P_1(x) = x$.

a) Prove that if $P_k(x)$ is the Legendre polynomial of degree k, then

$$\int_{-1}^{1} [P_k(x)]^2 dx = \frac{2}{2k+1}.$$

Use the three-term recurrence relation satisfied by Legendre polynomials.

b) Calculate the polynomial of degree ≤ 2 which minimizes

$$\int_{-1}^{1} [\sin \pi x - p(x)]^2 dx$$

over all polynomials $p(x)$ of degree ≤ 2. Use Legendre polynomials.

B10. Though it is possible to reduce the second order equation

$$y'' = f(x, y, y')$$

subject to the initial conditions $y(a) = \alpha_1$ and $y'(a) = \alpha_2$, to a standard first order system, it is sometimes more convenient to consider this equation directly when f is independent of y'. In such a case, consider the following solution method:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = \beta_0 f_{n+1} + \beta_1 f_n + \beta_2 f_{n-1},$$

where $y_n = y(x_n)$ and $f_n = f(x_n, y_n)$.

a) Find the values of β_0, β_1 and β_2 resulting in a method yielding the highest order of accuracy. What is the truncation error of the resulting scheme?

b) Using the model problem

$$y'' = -\lambda^2 y,$$

where λ is real and positive, obtain the stability restriction (if any) of the method corresponding to (a).

c) Give explicitly the initial conditions that would be used in obtaining the solution using the method obtained in (a).