APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

FALL 2003

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Fri., Sept. 4, 2003

Time: 12:00 – 4:00 PM

Place: Old Chemistry, Rm. 138

A1. Consider the following differential equation with a real parameter μ:

$$\begin{aligned} x' &= \mu x + y \\ y' &= x - 2y. \end{aligned}$$

a) Discuss the stability of the origin for $\mu = -2$.

b) Discuss the stability of the origin for $\mu = 2$.

c) Find all of the bifurcation points for the differential equation.

A2. Consider the boundary value problem with boundary conditions

$$\begin{aligned}
&\text{DE:} && Ly \equiv a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0, \qquad a \le x \le b \\
&\text{BC:} && B_a(y) \equiv \alpha_1\,y(a) + \beta_1\,y'(a) = A \\
& && B_b(y) \equiv \alpha_2\,y(b) + \beta_2\,y'(b) = B
\end{aligned}$$

a) Prove that the nonhomogenous problem ($A^2 + B^2 \neq 0$) has no solution or infinitely many solutions when the corresponding homogeneous problem ($A = B = 0$) has a non trivial solution.

b) Prove that this problem has a unique solution if and only if

$$\Delta = \begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix} \neq 0$$

where y_1 and y_2 are independent solutions of $Ly = 0$.

A3.

a) Derive a general solution to the 1D Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + m^2 u = 0$$

$$-\infty < x < \infty, \qquad 0 < t$$

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x)$$

where c and m are positive constants and $g(\cdot)$ and $h(\cdot)$ are assumed to have sufficient smoothness.

b) What relationship must exist between $g(x)$ and $h(x)$?

A4.

a) Obtain a formula for the solution for the heat equation in a semi-infinite rod,

$$u_t = u_{xx}, \qquad 0 < x, \quad 0 < t$$
$$u(x,0) = 10xe^{-x/2}$$

with the end point condition $u(0,t) = 0$.

b) Repeat, but for the end point condition $u_x(0,t) = 0$.

A5. Evaluate the integral

$$\oint_C \left(z + \frac{1}{z}\right)^n \, dz,$$

where n is a positive integer and C is any simple closed contour encircling $z = 0$.

B6. If each statement is true, provide a proof; otherwise disprove by providing a counter example.

a) A is an m by m matrix. Each column and each row of A contains one and only one 1 and zeros elsewhere. Then $\|A\|_p = 1$ for every $1 \leq p \leq \infty$.

b) B is an m by m matrix. Each column of B contains at most one 1 and zeros elsewhere. Then $\|B\|_p = 1$ for every $1 \leq p \leq \infty$.

B7.

Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

a) Determine a real SVD of A in the form $A = U\Sigma V^T$ where U and V are orthogonal matrices and $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ is a diagonal matrix with $\sigma_1 > \sigma_2$. The SVD is not unique, so find one that has the minimal number of minus signs in U and V.

b) Consider the linear map Ax for all vectors $x \in B_2$ where B_2 is the unit ball in $\mathbf{R}^2$. Draw a careful, labeled picture of the unit ball in $\mathbf{R}^2$ and its image under A, together with the singular vectors (column vectors of U and V), with the coordinates of their vertices marked. What is the geometric meaning of the singular values σ_1 and σ_2?

c) Find the 2-norm, $\|A\|_2$, and the Frobenius norm, $\|A\|_F$, from the SVD.

B8. Let $P_0(x)$, $P_1(x)$, $\cdots$ be a sequence of orthogonal polynomials and let $x_0, \cdots, x_k$ be the $k+1$ distinct zeros of $P_{k+1}(x)$. Prove that the Lagrange polynomials

$$l_i(x) = \Pi_{j \neq i}(x - x_j)/(x_i - x_j), \qquad i = 0, \cdots, k,$$

for these points are orthogonal to each other.

B9. Given a twice continuously differentiable function $f : [a, b] \to R$ and three points $x_0, x_1, x_2 \in [a, b]$ with $x_0 \neq x_2$, show that there exists a unique polynomial of degree 3 for which

$$p(x_0) = f(x_0), \quad p'(x_1) = f'(x_1), \quad p''(x_1) = f''(x_1), \quad p(x_2) = f(x_2).$$

Find a representation of the polynomial.

B10. Show that the one-step method for $y' = f(x, y)$ given by

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3),$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ k_3 &= f(x_n + h, y_n + h(-k_1 + 2k_2)), \end{aligned}$$

is of third order.