APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

FALL 2004

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ____________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., Sept. 1, 2004

Time: 12:00 – 4:00 PM

Place: Math Rm. 122-A

A1. Consider the boundary value problem

$$\begin{aligned} &y'' + \pi^2 y = f(t), \\ &A_1 y(0) + A_2 y'(0) = 0, \\ &B_1 y(1) + B_2 y'(1) = 0, \end{aligned}$$

where $f(t)$ is a continuous function on the interval $[0, 1]$.

a) Find conditions on A_1, A_2, B_1 and B_2 for which this problem has a unique solution.

b) Calculate the unique solution for the boundary value problem

$$y'' + \pi^2 y = \pi^2, \qquad y'(0) = 0, \quad y(1) = 0,$$

by first constructing the Green's function.

A2. Solve the initial value problem

$$\begin{cases} x' = x + y, & x(0) = 4, \\ y' = x + 2y, & y(0) = 2. \end{cases}$$

A3. Consider the wave equation

$$u_{tt} - u_{xx} = 0$$

in the domain $x \in (0,3)$ with the initial conditions

$$u(x,0) = \begin{cases} \sin \pi(x-1) & \text{if } 1 < x < 2, \\ 0 & \text{otherwise,} \end{cases}, \qquad u_t(x,0) = 0,$$

and boundary conditions

$$u(0,t) = 0, \qquad u_x(3,t) = 0.$$

Find and draw the solutions at $t = 1$, $t = 2$ and $t = 3$.

A4. Find the weak solution to the following conservation law problem

$$u_t + (u^2)_x = 0$$

with the initial condition

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases}$$

Consider both the cases $u_l > u_r$ and $u_l < u_r$. Find the solution at $x = 0$ in each case.

A5. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, \ldots, a_n, \ldots$ such that $f(a_n)$ is real for each n.

a) Prove that $f(x)$ is real for all real x.
[*Hint*: Show that $f(z) = \overline{f(\bar{z})}$ for all z.]

b) Moreover, if $a_1 > a_2 > \ldots > a_n > \ldots > 0$, with $\lim_{n\to\infty} a_n = 0$, and $f(a_{2n+1}) = f(a_{2n})$ for all n, prove that f must be a constant.
[*Hint*: Notice that $f'(x)$ must vanish somewhere inside each of the intervals $a_{2n+1} < x < a_{2n}$.]

B6. Consider constructing a matrix using a deck of playing cards. Treat A, J, Q, and K as 1, 11, 12 and 13, respectively and note that there are only 52 cards (4 cards for each number from 1 to 13). Let $M_p(m)$ be the maximum value for the matrix p-norm among all possible m by n matrices ($m \geq n$); let $M_F(m)$ be the maximum value of the Frobenius norm among all such m by n matrices.

a) Compute $M_1(6), M_1(10), M_\infty(6), M_\infty(10)$, and $M_F(3)$.

b) Compute $\max_{1 \leq m \leq 52} M_1(m)$ and $\max_{1 \leq m \leq 52} M_\infty(m)$.

B7. Let $A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$. Apply the inverse iteration with a shift $\mu = 2$ starting from $v^{(0)} = c \begin{pmatrix} \sqrt{3} - 2 \\ 3 - 2\sqrt{3} \end{pmatrix}$ where c is a normalization constant so that $v^{(0)}$ becomes a unit vector. Show that it finds an exact eigenvalue and an eigenvector in just one iteration. (You don't need to compute the constant c at all.)

B8. The midpoint and trapezoid integration rules have the same order of accuracy. Simpson's rule and the corrected trapezoid rule have the same order of accuracy. The accuracy of the latter two methods is greater than that of the first two. Using a single interval, evaluate the integral

$$I \equiv \int_0^1 xe^{-x}\, dx.$$

by one of the lower order methods AND THEN AGAIN by one of the higher order methods. For each of the two methods that you choose, obtain a bound on the error of the evaluation.

B9. The function $f(x) = x^5$ is tabulated at the points $x_i = i * h, \quad i \in Z$.

a) Determine the unique cubic polynomial $p_3(x)$ that interpolates $f(x)$ at an arbitrary real number $\bar{x}$ with minimum error at $\bar{x}$. Evaluate the coefficients of p_3 as concisely as you can.

b) Determine an upper bound for the minimum error at $\bar{x}$.

B10. Consider the general, first order, initial value problem

$$y' = f(x, y), \qquad y(x_0) \equiv y_0.$$

Design a Runge-Kutta scheme of the form

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2, \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1), \end{aligned}$$

that is exact through all second order terms AND the third order term f_{xx}. (Note: exactness in the other third order terms is not required.) Determine the leading order error term(s) for this scheme.