

**APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS**

Fall 1999

(CLOSED BOOK EXAM)

This is a two part exam.

In part A, solve 4 out of 5 problems for full credit.

In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME _____

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Fri., Sept. 3, 1999

Time: 12:30 – 4:30 PM

Place: Stony Brook Union, 231

A1. Let $u(t), v(t)$ be two solutions to the differential equation

$$\frac{dx}{dt} = f(x(t)),$$

on the closed interval $I := [t_0, t_1]$. Show that for all $t \in I$,

$$|u(t) - v(t)| \leq |u(t_0) - v(t_0)| \exp(\mu(t - t_0)),$$

where f satisfies the Lipschitz condition on I with the Lipschitz constant μ .

A2. Consider the coupled first-order differential equations

$$\frac{dx}{dt} = y(1 - x^2), \quad \frac{dy}{dt} = -x(1 - y^2).$$

- (a) Find all critical points of this system.
- (b) Discuss the stability of the linearized equations at the critical points found in part (a).
- (c) Find a first integral for this system. That is, find a function $F(x, y)$ such that $dF(x(t), y(t)) / dt = 0$ for solutions to this system.
- (d) Sketch the phase portrait of this system.

A3. Let u be a C^2 solution of the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

in the square $\Omega = \{ (x, y) : |x| < 1, |y| < 1 \}$. Assuming that u satisfies homogeneous Dirichlet boundary conditions, find the maximum and minimum values of u .

A4. Let $u \in C^2(\mathbf{R} \times [0, \infty))$ solve the following initial-value problem for the one-dimensional wave equation:

$$\begin{aligned}u_{tt}(x, t) - u_{xx}(x, t) &= 0 \quad \text{for } (x, t) \in \mathbf{R} \times (0, \infty) \\u(x, 0) &= g(x) \quad \text{for } x \in \mathbf{R}, \\u_t(x, 0) &= h(x) \quad \text{for } x \in \mathbf{R},\end{aligned}$$

where $g \in C^2(\mathbf{R})$ and $h \in C^1(\mathbf{R})$ have compact support. Let $K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t(x, t)^2 dx$ and $V(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x(x, t)^2 dx$ denote the kinetic and potential energies, respectively, of the solution at time $t \in [0, \infty)$.

- (a) Prove the principle of conservation of energy:
 $K(t) + V(t)$ is independent of t for all t .
- (b) Prove the principle of equipartition of energy:
 $K(t) = V(t)$ for all sufficiently large t .

A5. Prove that the function

$$f(z) = \frac{1}{\sin z}$$

has Laurent expansions of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

in each of the regions $\pi k < |z| < \pi(k+1)$, $k = 0, 1, \dots$

Find the coefficients $a_0, a_{-1}, a_{-2}, \dots$ of the expansion valid in the ring $\pi < |z| < 2\pi$.

B6. Show that

(a) $I - A^+A$ is a projector on the null space of A ,

(b) A^+A is a projector on the range of A^T ,

where A^+ is the Moore-Penrose generalized inverse of any matrix A .

B7. Given a symmetric, positive definite matrix A , identify a method to solve the system of linear equations $Ax = b$, where $x, b \in \mathbf{R}^n$ and determine the total number of multiplications and divisions needed in the solution. The method should utilize the properties of the matrix A .

B8. Describe the Successive Over-Relaxation method for solving $Ax = b$ and prove that the method converges if $0 < w < 2$, where w is the relaxation parameter.

B9. Given f_i and f'_i at the points x_i , $i = 1, 2$.

(a) Using Newton's divided difference formula, determine the cubic $P(x)$ such that

$$P(x_i) = f_i, \quad \text{and} \quad \frac{d}{dx}P(x_i) = f'_i.$$

(b) Show that

$$\int_{x_1}^{x_2} P(x) dx = (x_2 - x_1) \frac{f_1 + f_2}{2} + \frac{(x_2 - x_1)^2}{12} (f'_1 - f'_2).$$

(c) What is the numerical use of formulae such as that in part (b).

B10. Consider the initial-value problem

$$y'(x) = -\lambda y(x), \quad y(0) = 1, \quad \lambda > 0,$$

solved with the numerical method

$$y_n = y_{n-2} + \frac{1}{3}h [f(y_n) + 4f(y_{n-1}) + f(y_{n-2})] ,$$
$$y_0 = 1 .$$

What is the accuracy of this method? For small h , discuss the stability of the method. As $h \rightarrow 0$, to what solution does the numerical method converge?