Qualifying Exam (January 2009): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12,13,14,15,16.

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

1). Consider the problem $\{\min cx \mid Ax = b,\ l \le x \le u\}$ where l and u are finite vectors.

(a). Give the dual LP.

(b). Show that the dual always has a feasible solution.

(c). If the primal problem has a feasible solution, what conclusions would you reach?

2). Consider an LP (P) $\min\{cx \mid Ax = b,\ \ x \ge 0\}$, and its corresponding big M LP, called P(M)

(a). Prove that if $M_1 \le M_2$ then $z(M_1) \le z(M_2)$.

(b). Suppose that (P) has a finite optimal solution and denote it $z(\infty)$. Prove that $z(M) \le z(\infty)$.

(c). Show that there is a value M_0 such that for $M \ge M_0$, $z(M) = z(\infty)$, and so we can conclude that the big-M method will produce the right solution for large enough M.

3). Consider the following Linear Programming problem: $\min\{c^t x \mid Ax = b,\ x \ge 0\}$

(a). Suppose that (P) has at least one optimal solution. Let

$$S = \{i |\ x_i > 0 \quad \text{for at least one optimal solution, } x\}.$$

Prove that there exists an optimal solution x^* such that $x_i^* > 0$, for all $i \in S$, and $x_j^* = 0$, for all $j \notin S$.

(b). Assume that (P) has a non-degenerate basic feasible solution. Prove that there is a feasible solution, $\bar{x}$, such that $\bar{x} > 0$ (in other words, ALL components of the vector $\bar{x}$ are greater than zero).

For the next two parts, assume that (P) has been solved by the simplex method, yielding a finite optimum. Suppose it is now desired to add the constraint $\alpha x \le \beta$, and suppose the new program is feasible.

(c). Show that if x^0 is optimal for (P) and if $\alpha x^0 \le \beta$, then x^0 is optimal for the new problem.

(d). Suppose $\alpha x^0 > \beta$ for all optimal solutions, x^0, to (P). Show that if x^1 is optimal for the new problem, then $\alpha x^1 = \beta$.

4). Let S_i denote the time of the occurrence of the ith event in a Poisson process $\{N(t),\ t \ge 0\}$ with the intensity λ.

(a) Find $E[\frac{1}{N(t)+1}]$.

(b) Find $E[S_{N(t)}]$.

5). A machine shop needs a machine continuously. When a machine fails or it is 3 years old, it is instantaneously replaced by a new one. Successive machine lifetimes are i.i.d. random variables uniformly distributed over $[2, 5]$ years. The long-run rate of replacement is defined as the limit of the number of replacements by time t divided by t as $t \to \infty$. The long-run rate of planned replacement is the similar limit for the number of replacements occurred when the machine was in a working condition. Compute

(a) long-run rate of replacement.

(b) long-run rate of planned replacement.

6). Customers arrive according to a Poisson process to a service facility modelled as an M/M/1 queueing system. The arrivals decide independently whether to enter the facility or not. Each of them joins the facility with the probability $\frac{1}{n}$, where n is the number of the customers in the system that the arrival sees, and leaves without entering with the probability $\frac{n-1}{n}$. Find the long-range average number of customers in the system, if the service intensity is μ and the arrival intensity is λ.

7). A 95% confidence interval for the *difference* in average time in queue (Last Come First Served (LCFS) − First Come First Served (FCFS)) based on 1000 replications is 0.2 ± 0.4 min.

(a) Based on this confidence interval, which could you conclude is better, LCFS or FCFS? If you cannot make a conclusion, estimate the number of additional replications that should be run.

(b) Repeat (a) for the case where a variance reduction technique reduces variance to 25% of before, with the mean remaining unchanged.

8). Consider a single queue with the following data on customer interarrival and service times (in minutes) for 10 customers

Interarrival:	3	3	2	4	6	2	4	8	4	5	5
service times:	4.5	4	2	2	4	3	4	6	4	3	

We wish to simulate the number of customers in the system $N(t)$, assuming that the system is empty at time 0, i.e., $N(0) = 0$ and the first job arrives at $t = 3$. We consider a single server, first come first served (FCFS) queue discipline. If the terminating condition is completion of service for 10 customers, draw a sample path for $N(t)$ and compute the following performance measures:

(a) average time that a job waits in queue;

(b) fraction of jobs that spent more than 4.5 minutes in the system;

(c) fraction of time that the system is not idle;

(d) fraction of time that there is more than one customer in queue.

9).

(a). Given a set $\mathcal{C}$ of n unit disks in the plane (arbitrarily overlapping), explain briefly how to build a data structure to support efficient queries of the form: For a query (infinite) line L, determine if L stabs some disk of $\mathcal{C}$ and, if so, report one such disk. State the preprocessing time, storage space, and query time.

(b). Given a set P of n points in the plane, explain briefly how to build a data structure to support efficient queries of the form: For a query segment σ, known to be axis-parallel (i.e., horizontal or vertical), and a number $\delta > 0$ (given as part of the query), determine if there exists some point $p \in P$ whose Euclidean distance from σ is at most δ, and, if so, report one such point. (The distance from p to σ is defined to be $\inf_{q \in \sigma} d_2(p, q)$, where $d_2(\cdot, \cdot)$ denotes Euclidean distance.) State the preprocessing time, storage space, and query time.

10).

(a). For each of the computations below indicate how efficiently one can perform the calculation, in terms of $O(...)$ notation (e.g., $O(n)$, $O(\log n)$, $O(n^2)$, $O(n \log n)$, $O(k \log n)$, etc). Try to give the best (lowest) upper bound possible and give a brief justification of your answer.

(i). Given a set S of n line segments in the plane, each of length at least 1, determine whether or not there exist two lines, ℓ_1 and ℓ_2, that contain the segments (i.e., $S \subset \ell_1 \cup \ell_2$).

(ii). Given a set S of n points in $\Re^3$ (3D), determine if a particular triple of the points (say, $p, q, r \in S$) define a facet of the convex hull of S.

(b). For each of the following statements, state whether it is **ALWAYS TRUE**, **SOMETIMES TRUE** (but sometimes false), or **NEVER TRUE**. *JUSTIFY YOUR ANSWER, either with an example or with a brief explanation.* **SPECIFICALLY**, if it is sometimes true, sometimes false, DRAW examples of *each* (both true and false) case. If it is always true or always false, state why (briefly).

(i). Let S be a set of n points in the plane, not all on a line. Let T be a minimum spanning tree of S and let G be the nearest neighbor graph of S. Then the union of the edges from T and from G is the set of all Delaunay edges for S.

(ii). Let P be a simple polygon with (point) guard number $g(P)$. Then it is possible to guard P with at most $g(P) + 1$ guards whose visibility graph (within P) is connected (i.e., such that the graph that joins each pair of guards that see each other is a connected graph).

11). Recall, a strongly connected directed graph is a graph in which a (directed) path exists from every node i to every node j.

(a). Prove that every strongly connected graph on n nodes has a strongly connected subgraph on all n nodes containing at most $2(n-1)$ arcs.

(b). STRONGLY CONNECTED SUBGRAPH PROBLEM: Given a strongly connected directed graph $G = (N, A)$ and a bound K, is there a subset $A' \subset A$ with $|A'| \leq K$ such that $G' = (N, A')$ is strongly connected. Show that this problem is NP Complete.

12). A university has k clubs to which students may belong. (Each student can belong to many clubs.) We wish to form an organizing committe, which has a representative student from each club. The restriction is that a student must be a member of the club he/she represnts. A representative can also be a member of other clubs as well. However, each club must be represented by a different stud ent (i.e., the number of students in this organizing committee must be k).

(a). Show how to formulate the question of finding such a committe (deciding whether one exists) as a bipartite matching or max flow problem. Make sure to clearly define the graph!

(b). Give a necessary and sufficient condition for the existance of such an organizing committee.

13). Consider a Markov Decision Process (MDP) with the state space $X = \{1, 2, 3\}$ and the action sets $A(1) = A(2) = A(3) = \{a^1, a^2\}$. Let the transition probabilities be $p(x|y, a^1) = 1/3$, where $x, y \in X$, and $p(2|1, a^2) = p(3|1, a^2) = p(1|2, a^2) = p(3|2, a^2) = p(2|3, a^2) = p(3|3, a^2) = 1/2$. The one-step rewards are $r(1, a^1) = r(3, a^2) = 1$, $r(1, a^2) = r(2, a^1) = r(2, a^2) = 2$, and $r(3, a^1) = 3$. The goal is to maximize average rewards per unit time over the infinite horizon.

(a) Write the primal and dual linear programs (LPs) for this MDP.

(b) Explain how to compute an optimal policy from the optimal basic solution of the primal LP.

(c) Explain how to compute an optimal policy from the optimal basic solution of the dual LP.

14). Consider the following statement for a Markov Decision Process with a denumerable state space: any conserving stationary policy is optimal. Provide and prove your answers for the following two cases:

(a) positive dynamic programming;

(b) negative dynamic programming.

15). Let X be a random variable with $E[X] < 0$ and $E[e^{aX}] = 1$ for some constant a. Show that $a > 0$.

16). Provide an example of Riemann integrable functions f_n, $n = 1, 2, \ldots$, on $[0, 1]$ and a function f on the same set such that: (a) $f_n(x) \leq f(x)$ for all $x \in [0, 1]$ and for all $n = 1, 2, \ldots$, (b) $\lim_{n\to\infty} f_n(x) = f(x)$ μ-almost sure, where μ is the Lebesgue measure on $[0, 1]$, and (c) f is not Riemann integrable.