MATHEMATICAL STATISTICS
Fall 2000

NAME: ________________________________

The four problems you have attempted: ________________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let $X_1, X_2, \ldots$ are iid exponential(1).

   (a) Show that
   \[
   P \left( \frac{X_n - 1}{1/\sqrt{n}} \leq x \right) \rightarrow P(Z \leq x)
   \]
   for every $x$, where $Z$ is a standard normal random variable.

   (b) Show that differentiating both sides of the approximation in part (a) suggests
   \[
   \frac{\sqrt{n}}{\Gamma(n)} \left( x \sqrt{n} + n \right)^{n-1} \exp \left[ - (x \sqrt{n} + n) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \quad \text{as } n \to \infty.
   \]
   Hint: pdf of Gamma($\alpha, \beta$) distribution is
   \[
   f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \ x > 0; \ \alpha, \beta > 0.
   \]
2. Let \( X_1, X_2, \ldots, X_n, n \geq 3 \), be independent and identically distributed with

\[
P(X_1 = 1) = p = 1 - P(X_1 = 0).
\]

Consider unbiased estimation of \( g(p) = (1 - p)^2p \).

(a) Find the UMVUE of \( g(p) \).

(b) Derive a lower bound for the variance of any unbiased estimator.
3. Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, and suppose that the prior distribution on $\theta$ is $N(\mu, \tau^2)$. Here we assume that $\sigma^2$, $\mu$ and $\tau^2$ are all known.

(a) Find the posterior distribution $h(\theta|\overline{x})$ of $\theta$.

(b) Find the Bayes estimator of $\theta$ under the squared error loss.
4. Suppose random variables $Y_i, \ldots, Y_n$ satisfy

$$Y_i = \beta X_i + \epsilon_i, \ i = 1, \ldots, n,$$

where $X_1, \ldots, X_n$ are independent $N(\mu, \tau^2)$ random variables, and $\epsilon_1, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$, and the $X$'s and $\epsilon$'s are independent. Find approximate mean and variance for $\sum Y_i / \sum X_i$ in terms of $\mu, \tau^2$ and $\sigma^2$. 
5. Let $X_1, X_2, \ldots, X_n$ be iid with density $f_\theta(x) = \theta^{-1} \exp(-x/\theta)$ for $x > 0$, and denote the order statistics by $X(1) < X(2) < \cdots < X(n)$.

(a) Show that $Y_1, \ldots, Y_n$ are iid with density $f_\theta$, where $Y_i = (n - i + 1)[X(i) - X(i-1)]$ and $X(0) \equiv 0$. You may assume this result for part (b).

(b) Suppose you only get to observe $X(1), \ldots, X(k)$, where $k$ is a known constant. Starting from the Neyman-Pearson lemma, derive the UMP test for $H_0 : \theta \leq 1$ vs. $H_1 : \theta > 1$. What common tables can be used for finding the critical value of the test statistic? Justify your answer.
6. Let $X_1, \ldots, X_{n_1}$ be a random sample from a $N(\mu_1, \sigma^2)$ and let $Y_1, \ldots, Y_{n_2}$ be a random sample from a $N(\mu_2, \sigma^2)$ distribution which is independent of the first random sample. Consider the likelihood ratio test (LRT) for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

(a) Show that the LRT can be based on the statistic

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{S^2_p \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where

$$S^2_p = \frac{1}{n_1 + n_2 - 2} \left[ \sum_{i=1}^{n_1} (X_i - \bar{X})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2 \right].$$

(b) Show that, under $H_0$, $T \sim t_{n_1 + n_2 - 2}$. 