MATHEMATICAL STATISTICS QUALIFYING EXAM
Spring 2000

NAME: ______________________________

The four problems you have attempted: ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let X and Y be iid $N(0,1)$ random variables, and define $Z = \min(X, Y)$. Prove that $Z^2 \sim \chi_1^2$.

2. Let $X_1, X_2, \ldots$ be iid with the uniform distribution on $[0,\ 1]$. Find the number a so that

$$\sqrt{n}\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n} - a\right]$$

converges in distribution, and identify the limiting distribution.
(Hint: Define $Y_i = \log X_i$, $i = 1, \ldots, n$ and apply the CLT on $Y_1, \ldots, Y_n$ and the δ-method.)

3. Let S^2 be the sample variance based on a sample of size n from a normal population. The conjugate prior for σ^2 is the inverted gamma, $\mathrm{IG}(\alpha, \beta)$, with pdf

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha(\sigma^2)^{\alpha+1}} e^{-1/(\beta\sigma^2)}, \quad \sigma^2 > 0, \quad \alpha > 0, \beta > 0$$

with expectation $\mathrm{E}(\sigma^2) = 1/[(\alpha - 1)\beta]$. Find the Bayes estimator of σ^2 under the squared error loss.

4. Explain the error in the following argument:
Let $\delta(X)$ be any unbiased estimator of $g(\theta)$ and let T be any statistic (not necessarily sufficient). Then

$$\eta(t) = \mathrm{E}[\delta(X)|t]$$

is again an unbiased estimator of $g(\theta)$ and its variance is less than equal to that of $\delta(X)$.

5. Let $X_1, \ldots, X_n$ be a sample from the gamma distribution $\Gamma(a, b)$ with density

$$\frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}, \; x > 0, \; a > 0, \; b > 0.$$

Determine the UMP test for $H_0 : a \leq a_0$ against $a > a_0$ when b is known.

6. Let $X_1, \ldots, X_n$ be a random sample from Bernoulli(θ). Show that the likelihood ratio test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ will reject H_0 if $\sum_{i=1}^n X_i > c$ for some constant c.