MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 2001

NAME: ______________________________

The four problems you have attempted: ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let f be a mapping of the space $\boldsymbol{X}$ to the space $\boldsymbol{Y}$. Let $\{A_t : t \in T\}$ be a collection of subsets of $\boldsymbol{X}$. If we define for $A \subset \boldsymbol{X}$, $f(A) = \{f(x) : x \in A\}$, then

   (a) show $f\left(\bigcup_{t\in T} A_t\right) = \bigcup_{t\in T} f(A_t)$.

   (b) show $f\left(\bigcap_{t\in T} A_t\right) \subset \bigcap_{t\in T} f(A_t)$.

   (c) give an example where $f\left(\bigcap_{t\in T} A_t\right) \neq \bigcap_{t\in T} f(A_t)$.

2. If $X_1, X_2, \ldots$ are independent real valued random variables with common expectation μ and $\mathrm{E}(X_n^4)$ is bounded for all n, prove that

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty.$$

   The X_i are not necessarily identically distributed.

3. Observations $X_1, \ldots, X_n$ comprise a random sample from a Poisson distribution with mean θ. Define

$$\phi = \exp(-\theta).$$

   (a) When $n = 1$, find a UMVUE for ϕ and find the variance of this estimator.

   (b) For $n > 1$, find a UMVUE ϕ^* for ϕ together with the maximum likelihood estimator $\hat{\phi}$ of ϕ.

4. Let X be distributed as uniform on $(0, |\theta|^{-1})$, $1 \leq |\theta| < \infty$, and let the prior distribution have pdf $\lambda(\theta) = 1/(2\theta^2)$.

   (a) Show that the posterior pdf $h(\theta|x)$ is

$$h(\theta|x) = \begin{cases} \frac{1}{2|\theta|(-\log x)}, & \text{if } x \leq \frac{1}{|\theta|} \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Find the Bayes estimator of θ under the squared error loss.

5. Let X_i be independently distributed as $N(i\Delta, 1)$, $i = 1, \ldots, n$.

   (a) Show that there exists a UMP test of $H_0 : \Delta \leq 0$ against $H_1 : \Delta > 0$.

   (b) Determine the test as explicitly as possible.

6. Let $X_1, \ldots, X_{n_1}$ be a random sample from a $N(\mu_1, \sigma^2)$ and let $Y_1, \ldots, Y_{n_2}$ be a random sample from a $N(\mu_2, \sigma^2)$ distribution which is independent of the first random sample. Consider the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

   (a) Find the unrestricted MLE's of μ_1, μ_2 and σ^2.

   (b) Find the MLE's of $\mu = \mu_1 = \mu_2$ and σ^2 under H_0.

   (c) Show that the LRT can be based on the statistic

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2(\frac{1}{n_1} + \frac{1}{n_2})}},$$

   where

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1}(X_i - \bar{X})^2 + \sum_{i=1}^{n_2}(Y_i - \bar{Y})^2\right].$$