The four problems you have attempted: ________________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let $f$ be a mapping of the space $X$ to the space $Y$. Let $\{A_t : t \in T\}$ be a collection of subsets of $X$. If we define for $A \subset X$, $f(A) = \{f(x) : x \in A\}$, then
   
   (a) show $f(\bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(A_t)$.
   
   (b) show $f(\bigcap_{t \in T} A_t) \subset \bigcap_{t \in T} f(A_t)$.
   
   (c) give an example where $f(\bigcap_{t \in T} A_t) \neq \bigcap_{t \in T} f(A_t)$.

2. If $X_1, X_2, \ldots$ are independent real valued random variables with common expectation $\mu$ and $E(X_n^4)$ is bounded for all $n$, prove that

   $$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{p} \mu \quad \text{as} \quad n \to \infty.$$ 

   The $X_i$ are not necessarily identically distributed.

3. Observations $X_1, \ldots, X_n$ comprise a random sample from a Poisson distribution with mean $\theta$. Define

   $$\phi = \exp(-\theta).$$

   (a) When $n = 1$, find a UMVE for $\phi$ and find the variance of this estimator.

   (b) For $n > 1$, find a UMVE $\phi^*$ for $\phi$ together with the maximum likelihood estimator $\hat{\phi}$ of $\phi$.

4. Let $X$ be distributed as uniform on $(0, |\theta|^{-1})$, $1 \leq |\theta| < \infty$, and let the prior distribution have pdf $\lambda(\theta) = 1/(2\theta^2)$.

   (a) Show that the posterior pdf $h(\theta|x)$ is

   $$h(\theta|x) = \begin{cases} \frac{1}{2|\theta|(-\log x)}, & \text{if } x \leq \frac{1}{|\theta|} \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Find the Bayes estimator of $\theta$ under the squared error loss.

5. Let $X_i$ be independently distributed as $N(i\Delta, 1)$, $i = 1, \ldots, n$.

   (a) Show that there exists a UMP test of $H_0 : \Delta \leq 0$ against $H_1 : \Delta > 0$.

   (b) Determine the test as explicitly as possible.
6. Let \( X_1, \ldots, X_{n_1} \) be a random sample from a \( N(\mu_1, \sigma^2) \) and let \( Y_1, \ldots, Y_{n_2} \) be a random sample from a \( N(\mu_2, \sigma^2) \) distribution which is independent of the first random sample. Consider the likelihood ratio test for testing \( H_0 : \mu_1 = \mu_2 \) versus \( H_1 : \mu_1 \neq \mu_2 \).

(a) Find the unrestricted MLE’s of \( \mu_1, \mu_2 \) and \( \sigma^2 \).
(b) Find the MLE’s of \( \mu = \mu_1 = \mu_2 \) and \( \sigma^2 \) under \( H_0 \).
(c) Show that the LRT can be based on the statistic

\[
T = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}},
\]

where

\[
S_p^2 = \frac{1}{n_1 + n_2 - 2} \left[ \sum_{i=1}^{n_1} (X_i - \bar{X})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2 \right].
\]