1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Assume that $\sum_{n=1}^{\infty} \text{Var}(X_n)$ is finite. Show that $X_n$ converges to $E(X_n)$ in probability.
2. A bivariate population of \((X, Y)\) is sampled independently on three occasions. On the first, a random sample of size \(n_0\) is taken and only \(T = \min\{X, Y\}\) is observed for each pair. On the second, a random sample of size \(n_1\) is taken, and only the \(X\)-marginal is observed for each pair. Finally, a random sample of size \(n_2\) is taken, and only the \(Y\)-marginal is observed for each pair. Therefore, the combined set of observations is of the form \((T, X, Y)\), where \(T = (T_1, \ldots, T_{n_0})\), \(X = (X_{11}, \ldots, X_{1n_1})\) and \(Y = (Y_{21}, \ldots, Y_{2n_2})\). Assume the following two-parameter probability model for \((X, Y)\):

\[
P(X > x, Y > y) = \exp \left[ -\frac{1}{\theta} (x^{1/\delta} + y^{1/\delta})^\delta \right],
\]

\(x > 0, y > 0, \theta > 0, 0 < \delta \leq 1\) with unknown parameters \(\theta\) and \(\delta\).

(a) Present the joint pdf of \((T, X, Y)\).

(b) Is the above family an exponential family? Justify your answer.
3. Let $Y$ denote a random variable with a $\text{Binomial}(n, \theta)$ distribution. Suppose that the prior distribution of $\theta$ is $\text{Uniform}(0, 1)$.

(a) Find the posterior mean and standard deviation of $\theta$.

(b) Show that the posterior distribution of $\theta$ implies that $\phi = \theta/(1 - \theta)$ is distributed as a scale multiple of an $F$ random variable.

(Hint: If a random variable $U$ has a $\text{Beta}(\alpha, \beta)$ distribution, it is representable in the form $U = W_1/(W_1 + W_2)$, where $W_1$ and $W_2$ are independent $\chi^2$ random variables with respective degrees of freedom $2\alpha$ and $2\beta$.)
4. Let $X_1, \ldots, X_n$ be iid Gamma($\alpha, \beta$) with pdf

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \ x > 0; \ \alpha, \beta > 0.$$ 

Find the UMVUE of $1/\beta$ when $\alpha$ is known.
5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density $f(x; \theta)$. Determine the UMP test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ when the density $f(x; \theta)$ is Poisson($\theta$).
6. Let the random element $X$ be distributed as $P_\theta$ where $\theta \in \{\theta_0, \theta_1\}$. Let $\alpha$ and $\beta$ denote the type I and type II error probabilities, respectively, of a test for testing the simple null hypothesis $\theta_0$ versus the simple alternative $\theta_1$. Instead of the Neyman-Pearson approach of maximizing $1 - \beta$ subject to a fixed $\alpha$, another reasonable approach is to minimize a linear combination of $\alpha$ and $\beta$.

(a) Obtain the structure of a test $\phi^*(x)$, in terms of the likelihood ratio, that minimizes $2\alpha + \beta$. (Hint: The Neyman-Pearson lemma does not apply. Solve directly.)

(b) Prove that the same test $\phi^*(x)$ is also a most powerful level $\alpha_0$ test for some $0 \leq \alpha_0 \leq 1$. 