MATHEMATICAL STATISTICS QUALIFYING EXAM
Spring 2001

NAME: _____

The four problems you have attempted: _____

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Assume that $\sum_{n=1}^{\infty} \mathrm{Var}(X_n)$ is finite. Show that X_n converges to $\mathrm{E}(X_n)$ in probability.

2. A bivariate population of (X, Y) is sampled independently on three occasions. On the first, a random sample of size n_0 is taken and only $T = \min\{X, Y\}$ is observed for each pair. On the second, a random sample of size n_1 is taken, and only the X-marginal is observed for each pair. Finally, a random sample of size n_2 is taken, and only the Y-marginal is observed for each pair. Therefore, the combined set of observations is of the form $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{T} = (T_1, \ldots, T_{n_0})$, $\boldsymbol{X} = (X_{11}, \ldots, X_{1n_1})$ and $\boldsymbol{Y} = (Y_{21}, \ldots, Y_{2n_2})$. Assume the following two-parameter probability model for (X, Y):

$$P(X > x, Y > y) = \exp\left[-\frac{1}{\theta}(x^{1/\delta} + y^{1/\delta})^{\delta}\right],$$

$x > 0,\ y > 0,\ \theta > 0,\ 0 < \delta \leq 1$ with unknown parameters θ and δ.

(a) Present the joint pdf of $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$.

(b) Is the above family an exponential family? Justify your answer.

3. Let Y denote a random variable with a Binomial(n, θ) distribution. Suppose that the prior distribution of θ is Uniform$(0, 1)$.

   (a) Find the posterior mean and standard deviation of θ.

   (b) Show that the posterior distribution of θ implies that $\phi = \theta/(1-\theta)$ is distributed as a scale multiple of an F random variable.

   (Hint: If a random variable U has a Beta(α, β) distribution, it is representable in the form $U = W_1/(W_1 + W_2)$, where W_1 and W_2 are independent χ^2 random variables with respective degrees of freedom 2α and 2β.)

4. Let $X_1, \ldots, X_n$ be iid Gamma(α, β) with pdf

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \; x > 0; \; \alpha, \beta > 0.$$

Find the UMVUE of $1/\beta$ when α is known.

5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density $f(x;\theta)$. Determine the UMP test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ when the density $f(x;\theta)$ is Poisson(θ).

6. Let the random element X be distributed as P_θ where $\theta \in \{\theta_0, \theta_1\}$. Let α and β denote the type I and type II error probabilities, respectively, of a test for testing the simple null hypothesis θ_0 versus the simple alternative θ_1. Instead of the Neyman-Pearson approach of maximizing $1 - \beta$ subject to a fixed α, another reasonable approach is to minimize a linear combination of α and β.

   (a) Obtain the structure of a test $\phi^*(x)$, in terms of the likelihood ratio, that minimizes $2\alpha + \beta$. (Hint: The Neyman-Pearson lemma does not apply. Solve directly.)

   (b) Prove that the same test $\phi^*(x)$ is also a most powerful level α_0 test for some $0 \leq \alpha_0 \leq 1$.