MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 2002

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let $X_1, X_2, \ldots$ be a sequence of iid random variables with a common uniform distribution on (0, 1).

   (a) Let $Z_n = (\prod_{i=1}^n X_i)^{1/n}$ be the geometric mean of $X_1, \ldots, X_n$. Show that Z_n converges in probability to a constant c as $n \to \infty$ and find c.

   (b) Let $W_n = \max\{X_1, \ldots, X_n\}$ and show that W_n converges in probability to 1 as $n \to \infty$.

2. A random variable X possesses a <u>skewed-normal distribution</u> with mode μ and scale parameters σ_1^2 and σ_2^2 if its density is given by

$$f\left(x|\mu, \sigma_1^2, \sigma_2^2\right) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2} (\sigma_1 + \sigma_2)^{-1} \exp\left\{-\frac{1}{2}\sigma_1^{-2}(x-\mu)^2\right\} & \text{for } -\infty < x \le \mu \\ \left(\frac{2}{\pi}\right)^{1/2} (\sigma_1 + \sigma_2)^{-1} \exp\left\{-\frac{1}{2}\sigma_2^{-2}(x-\mu)^2\right\} & \text{for } \mu < x < \infty. \end{cases}$$

   (a) Show that

$$P(X \le \mu) = \frac{\sigma_1}{\sigma_1 + \sigma_2} \quad \text{and} \quad P(X > \mu) = \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

   (b) Show that

$$\mathrm{E}(X) = \mu + \left(\frac{2}{\pi}\right)^{1/2} (\sigma_2 - \sigma_1).$$

3. Let $X_1, \ldots, X_n$ be iid random variables with

$$P_\theta(X_j = 1) = \theta \quad \text{and} \quad P_\theta(X_j = 0) = 1 - \theta, \quad j = 1, \ldots, n,$$

   where n is a fixed constant and $0 < \theta < 1$. Let

$$\overline{X} = \frac{1}{n} \sum_{j=1}^n X_j.$$

   (a) Show that $\overline{X}$ is a complete sufficient statistic.

   (b) Show that an estimator $T = T(X_1, \ldots, X_n)$ is a UMVUE of its expectation if and only if it is a function of $\overline{X}$.

   (c) Show that a function $g(\cdot)$ of θ has an unbiased estimator if and only if $g(\cdot)$ is a polynomial of degree less than or equal to n.

   (d) Let T_k denote the UMVUE of θ^k. Give an explicit formula for T_k, $k = 1, \ldots, n$.

4. An observation y is normally distributed with unknown mean θ and known variance τ^2.

   (a) Before observing y, you have prior probabilities ϕ, that $\theta = \theta_0$, and $1 - \phi$, that $\theta = \theta_1$, where θ_0 and θ_1 are specified. Find your posterior probability that $\theta = \theta_0$, after observing y.

   (b) Before observing y, you have prior probabilities ϕ, that $\theta = \theta_0$, and $1 - \phi$, that $\theta \neq \theta_0$. Given that $\theta \neq \theta_0$, your prior distribution on θ is normal, with known mean θ_1 and variance σ^2. Find your posterior probability that $\theta = \theta_0$, after observing y.

5. Assume $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent random samples of size n from $N(\mu_1, 1)$ and $N(\mu_2, 1)$, respectively. Further, it is known that $\mu_2 \leq \mu_1$. Under this ordering of the parameters μ_1 and μ_2, answer the following questions.

   (a) Derive the maximum likelihood estimator of (μ_1, μ_2).

   (b) Derive λ, the likelihood ratio test statistic for testing H_0: $\mu_1 = \mu_2$ versus H_1: $\mu_1 > \mu_2$, and indicate the rejection region for the likelihood ratio test.

   (c) What is the distribution of $-2 \ln \lambda$ when $\mu_1 = \mu_2$?

6. Consider independent Bernoulli random variables X_i which have known covariate values (constants) z_i, $i = 1, 2, \ldots, n$, such that

$$P(X_i = 0) = \frac{1}{1 + \exp(\mu + \theta z_i)} = P_{i0} \quad \text{and} \quad P(X_i = 1) = \frac{\exp(\mu + \theta z_i)}{1 + \exp(\mu + \theta z_i)} = 1 - P_{i0},$$

   where (μ, θ) are unknown.

   (a) Determine a complete sufficient statistic for (μ, θ).

   (b) Derive the form of the UMP unbiased test of the hypothesis H_0: $\theta \leq 0$ versus H_1: $\theta > 0$. An explicit expression for the exact "constants" required to carry out the test is not readily available and hence not required, but you should give some indication of how the test would be carried out in practice.