MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 2003

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let X_n and Y_n, $n \geq 1$, be random variables, all defined on the same probability space. Suppose the Y_n's converge to 0 in probability as $n \to \infty$, and suppose the X_n's are bounded in probability (that is, for any $\epsilon > 0$, there is a number $M_\epsilon < \infty$ such that $P(|X_n| \geq M_\epsilon) < \epsilon$ for all n). Show that the products $X_n Y_n$ converge to 0 in probability as $n \to \infty$.

2. Let $X_1, X_2, \ldots$ be independent random variables with $X_k \sim \text{Poisson}(\lambda_k)$, $\lambda_k = 2 + \sqrt{k-1} - \sqrt{k}$. Find the limiting distribution of $\sqrt{n}(\overline{X}_n - 2)$, where $\overline{X}_n = n^{-1} \sum_{i=1}^n X_i$.

3. Let X be a discrete random variable with pdf
$$f_\theta(x) = \binom{r+x-1}{x} \theta^x (\theta+1)^{-(r+x)}, \quad x = 0, 1, 2, \ldots,$$
where r is a known positive integer and $\theta > 0$ is unknown. (This is the negative binomial distribution reparametrized so that $\theta = (1-p)/p$, where p is the probability of success on an individual trial.) The mean and variance of X are $\text{E}(X) = r\theta$ and $\text{Var}(X) = r\theta(\theta+1)$. We are interested in estimating θ using the loss function $L(\theta, \delta) = (\theta - \delta)^2/[\theta(\theta+1)]$. Find the Bayes estimator of θ with respect to the prior distribution with pdf
$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (\theta+1)^{-(\alpha+\beta)}, \ \theta > 0.$$

4. The random variable X has a mixture distribution: with probability π, it is 0; and with probability $1-\pi$, it is distributed as a chi-squared random variable which ν degrees of freedom. The probability density function of a chi-squared random variable with ν degrees of freedom is
$$f(x|\nu) = \frac{1}{\Gamma(\nu/2) 2^{\nu/2}} x^{(\nu/2)-1} e^{-x/2}, \quad 0 < x < \infty, \ \nu = 1, 2, \ldots.$$
A random sample of n observations is taken from X. Find the maximum likelihood estimates of π and ν.

5. The random variable Y has a geometric distribution with parameter π; that is,
$$P(Y = y|\pi) = \pi(1-\pi)^{y-1}; \ y = 1, 2, \ldots; \ 0 < \pi < 1.$$
Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from Y. Determine whether or not a UMP test of H_0: $\pi \geq \pi_0$ versus H_1: $\pi < \pi_0$ exists. If it does, find the UMP.

6. The random variable X has a Poisson distribution conditional on $X > 0$; that is,

$$P(X = x|\mu) = \frac{e^{-\mu}\mu^x}{(1 - e^{-\mu})x!}; \; x = 1, 2, \ldots; \; \mu > 0.$$

A random sample of size n is taken from X. Calculate the asymptotic variance of the maximum likelihood estimator of μ.