MATHEMATICAL STATISTICS QUALIFYING EXAM
Spring 2003

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let f_1 and f_2 be pdf's with $f_1(x) > 0$ and $f_2(x) > 0$ for $-\infty < x < \infty$. Show that the Kullback information number

$$I = \int_{-\infty}^{\infty} \log\left\{\frac{f_1(x)}{f_2(x)}\right\} f_1(x)dx$$

is always non-negative.

2. Let $X_1, X_2, \ldots$ denote a sequence of binomial random variables, such that, for $n = 1, 2, \ldots$, X_n has a binomial distribution with probability $1/n$ and sample size n. Show, for $k \geq 0$, that

$$\lim_{n\to\infty} \mathrm{E}(X_n^k) = e^{-1}\sum_{x=0}^{\infty}\frac{x^k}{x!}.$$

Is this result necessarily true for $k < 0$?

3. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ. Then $\phi = \exp(-\theta)$ denotes the probability that any particular observation is zero.

   (a) When $n = 1$, find a UMVUE for ϕ and show that this estimator has variance $\phi(1-\phi)$.

   (b) For $n > 1$, find a UMVUE ϕ^* for ϕ together with the MLE $\hat{\phi}$ of ϕ.

4. Let X be a Poisson random variable with mean $\lambda > 0$. Suppose we use the Gamma prior

$$f(\lambda) = \frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, \quad \lambda > 0$$

where $\alpha > 1$ and $\beta > 0$. Find the Bayes estimator of λ under loss

$$L(\lambda, d) = \frac{(\lambda - d)^2}{\lambda}.$$

5. Let $X_1, \ldots, X_{n_1}$ denote a random sample from the Gamma$(\alpha, 1/\beta)$ distribution, with density

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}\exp(-x/\beta)}{\beta^{\alpha}\Gamma(\alpha)}, \quad x > 0$$

where α and β are unknown positive parameters. Let $Y_1, \ldots, Y_{n_2}$ denote an independent random sample from the Gamma$(2\alpha, 1/\beta)$ distribution with density

$$f(y|\alpha,\beta) = \frac{y^{2\alpha-1}\exp(-y/\beta)}{\beta^{2\alpha}\Gamma(2\alpha)}, \quad y > 0.$$

Use all the data to develop a UMPU test for $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha > \alpha_0$.

6. A random sample, $X_1, \ldots, X_n$, is drawn from a Pareto population with pdf

$$f(x|\theta,\nu) = \frac{\theta\nu^{\theta}}{x^{\theta+1}} I_{[\nu,\infty)}(x), \;\; \theta > 0, \;\; \nu > 0.$$

(a) Find the MLEs of θ and ν.

(b) Suppose ν is unknown. Show that the likelihood ratio test of

$$H_0 : \theta = 1 \text{ versus } H_1 : \theta \neq 1$$

has critical region of the form $\{\boldsymbol{x} : T(\boldsymbol{x}) \leq c_1 \text{ or } T(\boldsymbol{x}) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log\left\{\frac{\prod_{i=1}^{n} X_i}{(X_{(1)})^n}\right\}.$$