MATHEMATICAL STATISTICS QUALIFYING EXAM
Spring 2004

NAME: ____________________ **ID:** ____________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________

1. Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with zero mean and variance ϕ. Let $S = \sum_{i=1}^n Y_i^2$ and $T_n = \log(S/n)$. Find the limiting distribution of

$$U = \sqrt{n}(T_n - \log\phi)$$

as $n \to \infty$.

2. Let (U, V) be a random vector having a bivariate exponential distribution with joint pdf

$$f(u,v) = \begin{cases} \frac{(\alpha+\beta)\beta}{2} e^{-\alpha u - \beta v} & \text{if } u < v \\ \frac{(\alpha+\beta)\beta}{2} e^{-\beta u - \alpha v} & \text{if } u > v \end{cases}, \quad \alpha > 0, \ \beta > 0.$$

   (a) Show that the random variables $X = \min\{U, V\}$ and $Y = |U - V|$ are independent, and find the marginal distributions of X and Y.

   (b) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random samples from exponential distributions with pdfs

$$f_X(x) = \lambda_1 e^{-\lambda_1 x}, \ x > 0, \quad f_Y(y) = \lambda_2 e^{-\lambda_2 y}, \ y > 0.$$

   Find the MLEs of λ_1 and λ_2 under the condition that we know $\lambda_1 \geq \lambda_2$. Relate this problem to the problem of estimating the parameters α and β from a random sample of size n from the distribution in (a).

3. Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, $\epsilon_1, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$, and σ^2 is unknown.

   (a) Find the MLE of β.

   (b) Find the exact distribution of the MLE of β.

   (c) Find the exact distribution of $\sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$.

   (d) Compare the variances of $\sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$ and the MLE of β.

4. Let X and Y have the joint distribution determined by

$$P(Y = y) = p^y(1-p)^{1-y}, \quad y = 0, 1$$

and

$$f(x|y) = (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{(x-\mu_y)^2}{2\sigma^2}\right], \quad -\infty < x < \infty.$$

The parameters satisfy $0 < p < 1$, $\sigma^2 > 0$ and $-\infty < \mu_0 < \infty$, $-\infty < \mu_1 < \infty$.
If (X_i, Y_i), $i = 1, \ldots, n$ are independent random vectors with the above joint distribution for (X_i, Y_i), find the UMVUEs for

(a) p

(b) $\mu = \mu_1 p + \mu_0(1-p)$

(c) σ^2

5. Let Y_1 and Y_2 denote independent random variables where, for $i = 1, 2$, Y_i has a binomial distribution with probability θ_i and sample size n_i. Suppose that the prior distribution of (θ_1, θ_2) is uniform over the unit square $(0,\ 1)\times(0,\ 1)$.

(a) Find the posterior mean and standard deviation of $\theta_1 - \theta_2$.

(b) Show that the posterior distribution of (θ_1, θ_2) implies that

$$\phi = \frac{\theta_1/(1-\theta_1)}{\theta_2/(1-\theta_2)}$$

is distributed as a scale multiple of the ratio of two independent F-random variables. (Hint: If a random variable U has a Beta(α, β) distribution, it is representable in the form $U = W_1/(W_1 + W_2)$, where W_1 and W_2 are independent χ^2 random variables with respective degrees of freedom 2α and 2β.)

(c) Explicitly express the posterior mean of $\log\phi$ in terms of the digamma function $\psi(t) = \partial \log\Gamma(t)/\partial t$ using the fact that the pdf of a $\chi^2_{2\alpha}$ random variable is

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)2^\alpha} e^{-x/2}, \quad x > 0.$$

6. Suppose a discrete random variable X takes on values $x = 1, 2, 3$ or 4 with probabilities $p_\theta(x)$ given in the table below, where the parameter $\theta = 0, 1$ or 2. Show that there exists a UMP level-α test of $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$ or 2 if $\alpha \leq .1$, but not for any larger α except $\alpha = 1$.

x	1	2	3	4
$p_0(x)$	.1	.2	.3	.4
$p_1(x)$	.4	.1	.2	.3
$p_2(x)$	.3	.4	.1	.2