MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 2006

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. If X_n has a Poisson distribution with parameter $\lambda = n$, $n = 1, 2, \ldots$, prove that

$$Z_n = \frac{X_n - n}{\sqrt{n}} \xrightarrow{d} N(0, 1)$$

   (a) using a moment generating function approach.

   (b) using the central limit theorem.

   Hint: Use the following formulae if you need to:

   - The mgf of Poisson(λ): $M_X(t) = \exp[\lambda(e^t - 1)]$
   - The mgf of $N(\mu, \sigma^2)$: $M_X(t) = \exp[\mu t + \sigma^2 t^2/2]$

2. Suppose that an observation X possesses a normal distribution with unknown mean θ and known variance τ^2 and consider the following prior density, defined for $-\infty < \theta < \infty$, for θ.

$$\pi(\theta) = \begin{cases} K \exp\left\{-\frac{1}{2}\phi^{-1}(\theta-\mu)^2 + \frac{1}{2}\phi^{-1}\xi^2\right\} & \text{for } |\theta-\mu| < \xi \\ K \exp\left\{-\frac{1}{2}\lambda^{-1}(\theta-\mu)^2 + \frac{1}{2}\lambda^{-1}\xi^2\right\} & \text{for } |\theta-\mu| \geq \xi \end{cases}$$

where K is a constant which ensures that $\pi(\theta)$ integrates to unity, and μ, ξ, ϕ and λ are specified prior parameters, with $\lambda \geq \phi$.

(a) Find the corresponding posterior density for θ.

(b) The posterior density for θ possesses a local maximum at either one or both of the points

$$\theta_1 = \frac{\tau^{-2}x + \phi^{-1}\mu}{\tau^{-2} + \phi^{-1}} \quad \text{and} \quad \theta_2 = \frac{\tau^{-2}x + \lambda^{-1}\mu}{\tau^{-2} + \lambda^{-1}}.$$

Obtain and carefully describe the posterior mode of θ, i.e., the global maximum of the posterior density.

3. Let $Y_1, \ldots, Y_n$ be independent normal random variables such that for $i = 1, \ldots, n$, $\mathrm{E}(Y_i|X = x_i) = \beta x_i$ and $\mathrm{Var}(Y_i|X = x_i) = \sigma^2 w_i^{-1}$, $w_i > 0$. If the conditional distribution of Y given x is Poisson, and $w_i^{-1} = x_i$, prove that the MLE of β is also the weighted least squares estimator obtained by minimizing $\sum_{i=1}^n w_i(y_i - \beta x_i)^2$.

4. Let $P(X_i = k) = 1/(2M + 1)$ for $k = 0, \pm 1, \pm 2, \ldots, \pm M$, where M is an unknown whole number.

   (a) Show that the complete sufficient statistic for M is $\max\{|X_1| \ldots, |X_n|\}$.

   (b) Derive the UMVUE of M based on an independent sample $X_1, \ldots, X_n$.

5. Let the distribution of X be given by

x	0	1	2	3
$P_\theta(X = x)$	θ	2θ	$0.9 - 2\theta$	$0.1 - \theta$

where $0 < \theta < 0.1$. For testing H_0: $\theta = 0.05$ against H_1: $\theta > 0.05$ at level $\alpha = 0.05$, determine which of the following tests (if any) is UMP.

(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$

(ii) $\phi(1) = 0.05$, $\phi(0) = \phi(2) = \phi(3) = 0$

(iii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$

6. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ).

   (a) Find the likelihood ratio test of H_0: $\theta = \mu$ versus H_1: $\theta \neq \mu$.

   (b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i}.$$

   (c) Find the distribution of T under H_0.