MATHEMATICAL STATISTICS QUALIFYING EXAM
Spring 2006

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. If $\{A_n, n \geq 1\}$ is an increasing sequence of events, show that

$$\lim_{n \to \infty} P(A_n) = P\left(\lim_{n \to \infty} A_n\right).$$

2. Let $Y_{(n)}$ denote the nth order statistic of a random sample of size n from a uniform distribution on the interval $(0, \theta)$. Prove that

$$Z_n = \sqrt{Y_{(n)}} \xrightarrow{P} \sqrt{\theta}.$$

3. Let $X_1, \dots, X_n$ be iid $N(\theta, 1)$.

   (a) Find the UMVUE of θ^2.

   (b) Find the Cramér-Rao Lower Bound for estimating θ^2.

4. (a) If $Y_1, \ldots, Y_n$ is a random sample from over-dispersed Poisson(μ) with over-dispersion parameter ϕ, then $\mathrm{E}Y_i = \mu$ and $\mathrm{Var}(Y_i) = \mu(1 + \phi\mu)$. In Poisson regression, the Anscombe residual is defined as

$$R(Y_i, \mu) = \frac{t(Y_i) - \mathrm{E}[t(Y_i)]}{\sqrt{\mathrm{Var}[t(Y_i)]}}, \quad i = 1, \ldots, n,$$

where $t(x) = x^{2/3}$. Find $\mathrm{E}[t(Y_i)]$ and $\mathrm{Var}[t(Y_i)]$ asymptotically, and express $R(Y_i, \mu)$ asymptotically in terms of Y_i, μ and ϕ.

(b) Suppose the survival time (in week) of a certain strain of mice has a distribution with cdf

$$F(t) = 1 - \exp\left[-\eta\delta_1(t/104)^{\delta_2}\right],$$

where η, δ_1 and δ_2 are constants. Show how to generate a random number from this distribution starting with a Uniform random number. Derive an explicit formula.

5. Let $X_1, \dots, X_n$ be a random sample of size n from Poisson(θ).

(a) Find the maximum likelihood estimator of θ.

(b) Assume the prior distribution on θ is Gamma$(a, 1/c)$ with pdf

$$\lambda(\theta) = \frac{c^a}{\Gamma(a)} \theta^{a-1} e^{-c\theta}, \theta > 0.$$

Find the Bayes estimator of θ under a squared error loss.

6. Let X_i be independently distributed as $N(i\Delta, 1),\ i = 1, \ldots, n$.

(a) Show that there exists a UMP test of $H_0 : \Delta \leq 0$ against $H_1 : \Delta > 0$.

(b) Find the distribution of the test statistic obtained in part (a) under H_0.
Hint: $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

(c) Determine the test as explicitly as possible.