MATHEMATICAL STATISTICS QUALIFYING EXAM

Fall 2007

NAME: ______________________________ **ID:** ______________________________

Instruction: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let $X_1, ..., X_n$ be an iid sample from a distribution with density function

$$f(x;\lambda) = \frac{x}{\lambda}\exp\left[-x^2/(2\lambda)\right],$$

where the parameter λ is known to be positive and $x > 0$ as well.

   (a) Find the MLE for λ.

   (b) Calculate $\mathrm{E}_\lambda(X_i^2)$. *Hint*: Do not integrate by parts; instead, use the fact that the derivative with respect to λ of the loglikelihood has mean 0.

   (c) Calculate $I(\lambda)$, the Fisher information for one observation about λ.

   (d) Is the MLE a sufficient statistic?

2. (*How to ask embarrassing questions*) You are asked to estimate the percentage of students on campus who have ever cheated, but you do not want to introduce a bias into your sample due to some students' reluctance to tell the truth. Tell each person sampled to secretly flip a fair coin. If the person gets a head, the person answers the question: *Have you ever cheated?* Otherwise, the person answers the innocuous questions: *Were you born in New York?* The information you get from each response is a yes or no (and not the outcome of the coin toss). Furthermore, you know from admission records that the proportion of students on campus born in New York is 0.6.

   (a) Let θ denote the proportion of students on campus who have ever cheated. As a function of θ, what is the probability that someone sampled responds with a yes?

   (b) Let X denote the number of yes responses in your sample of size n. Suggest an estimator of θ (Your answer should be a function of X and n).

   (c) Find the variance of the estimator.

   (d) Find an approximate 95 percent confidence interval for θ based on a sample of 100 with a half responding yes.

3. Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that

(a) $X_n \to 0$ in probability if and only if $p_n \to 0$.

(b) $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

4. Toss a coin with unknown probability p of landing heads. Toss it until you observe exactly 6 heads, and let X be the total number of tosses you made.

   (a) What is $P(X = j)$ as a function of p and j?

   (b) For testing a null hypothesis H_0: $p = 1/2$ versus the alternative hypothesis H_1: $p > 1/2$, determine if there exists a uniformly most powerful test at level $\alpha = 1/16$. If it exists, find it. Would you reject if $X = 10$?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Poisson(θ).

(a) Find the maximum likelihood estimator of θ.

(b) Assume the prior distribution on θ is Gamma($a, 1/c$) with pdf

$$\lambda(\theta) = \frac{c^a}{\Gamma(a)} \theta^{a-1} e^{-c\theta}, \theta > 0.$$

Find the Bayes estimator of $1/\theta$ under a squared error loss.

6. Suppose it is known that a new treatment is successful in curing a muscular pain with probability p. The treatment is independently tried on n patients. By answering the following questions, estimate the probability that one patient is cured and two other patients are not cured.

Let $X_1, X_2, \ldots, X_n,\ n \geq 3$, be iid with

$$P[X_1 = 1] = p = 1 - P[X_1 = 0].$$

Consider unbiased estimation of $g(p) = p(1-p)^2$.

(a) Construct an unbiased estimator $U(X_1, X_2, X_3)$ based on X_1, X_2, X_3.

(b) Find a complete sufficient statistic.

(c) Apply the Rao-Blackwell technique to U to obtain the UMVUE.

(d) Derive a lower bound for the variance of any unbiased estimator.