MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 1998

1. Let $\Omega_0 \subset \Omega$, $\mathbf{C}$ a Borel field containing Ω, and let $\mathbf{C}_0 = \{A_0, A_0 = A \cap \Omega_0, A \in \mathbf{C}\}$. Show that $\mathbf{C}_0$ is a Borel field containing Ω_0.

2. Show that if X is integrable and thus $\mathrm{E}(|X|)$ is finite,

$$\mathrm{E}(|X|) = \int_0^\infty [1 - F(x) + F(-x)]dx.$$

3. Let $X_n, Y_n,\ n \geq 1$ be random variables, all defined on the same probability space. Suppose the Y_n's converge to 0 in probability as $n \to \infty$, and suppose the X_n's are bounded in probability (that is, for any $\epsilon > 0$, there exists an $M_\epsilon < \infty$ such that $P(|X_n| \geq M_\epsilon) < \epsilon$ for all n). Show that the products $X_n Y_n$ converge to 0 in probability as $n \to \infty$.

4. Let $X_1, \ldots, X_n$ be independently and identically distributed according to a continuous cdf $F(x)$. It is sometimes desirable to determine a random lower limit $L(X_1, \ldots, X_n)$ that has, with probability at least $1-\alpha$, at least proportion β of the population to the right. Then $L(X_1, \ldots, X_n)$ is called a *one-sided nonparametric tolerance limit.* That is,

$$1-\alpha \leq P[L(X_1, \ldots, X_n) \leq \xi_{1-\beta}] = P[F(L(X_1, \ldots, X_n)) \leq 1-\beta].$$

Find the smallest sample size such that the smallest order statistic is the tolerance limit.

5. Let $X_1, \ldots, X_n$ be iid each with known probability p as $N(\mu, 1)$ and probability $1-p$ as $N(\mu, \sigma^2)$. Show that no MLE exists for $\theta \in \Theta = \{-\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$. (Hint: Show that the likelihood function approaches ∞ for some choice of μ and σ^2.)

6. A random variable X has the gamma distribution $\Gamma(\alpha, b)$ whose density is

$$\frac{1}{\Gamma(\alpha)b^\alpha} x^{\alpha-1} e^{-x/b}, \quad x > 0, \ \alpha > 0, \ b > 0.$$

(a) Show $\mathrm{E}(X^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} b^r$ for any integer r.

(b) Let $X_1, \ldots, X_n$ be iid according to $N(0, \sigma^2)$, so that the joint density of the X_i's is

$$c\tau^r \exp(-\tau \sum x_i^2),$$

where $\tau = 1/(2\sigma^2)$, $r = n/2$, and c is a constant. Suppose the prior distribution of τ be $\Gamma(g, 1/\alpha)$. Find the Bayes estimator of σ^2 under the squared error loss.

7. Let $X_1, \ldots, X_n$ be a random sample from $N(\xi, \sigma^2)$ with known ξ.

(a) Show that $S^2 = \sum_{i=1}^n (X_i - \xi)^2$ is a complete sufficient statistic for σ^2.

(b) Find the UMVUE of σ^r for any positive integer r.
Hint: Use the following result without proof.
If a random variable Y has the χ^2 distribution with degrees of freedom n, then

$$\mathrm{E}(Y^{r/2}) = \frac{\Gamma[(n+r)/2]}{\Gamma(n/2)} 2^{r/2}.$$

8. Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{[1+e^{(x-\theta)}]^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) Based on one observation, X, find the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

(b) For $\alpha = .2$, find the power of this test.

(c) Show that the test in part (a) is UMP size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.