1. Let \( \{A_n\} \) be a sequence of sets and \( P \) a probability function.

   (a) Show
   \[
P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i).
   \]

   (b) Show the Bonferroni’s inequality,
   \[
P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n - 1).
   \]
2. Let $F_{m,n}$ denote a random variable having the $F$ distribution with $m$ and $n$ degrees of freedom in the numerator and denominator, respectively. Let $\chi^2_m$ denote a chi-squared random variable with $m$ degrees of freedom. Show that $n F_{n,m}$ converges to $\chi^2_n$ as $m \to \infty$, where $n$ is fixed.
3. Let \( \{X_1, X_2, \ldots\} \) denote a sequence of independent and identically distributed random variables with zero mean and variance \( \sigma^2 < \infty \). Show that

\[
\lim_{n \to \infty} E(|X_1 + \cdots + X_n|/\sqrt{n}) = (2/\pi)^{1/2}\sigma.
\]

(Hint: Suppose that \( \{Y_1, Y_2, \ldots\} \) is a sequence of non-negative random variables such that

(a) For \( p > 1 \), \( \sum_{n=1,2,\ldots} E(Y_n^p) < c \) for some constant \( c \),

(b) \( Y_n \) converges in distribution to a random variable \( Y \).

Then \( E(Y) \) is finite, and \( \lim_{n \to \infty} E(Y_n) = E(Y) \). You don’t need to show this.)
4. Let $X_1, \ldots, X_n$ be iid random variables with

$$P_\theta(X_j = 1) = \theta = 1 - P_\theta(X_j = 0), \quad j = 1, \ldots, n,$$

where $n$ is a fixed constant and $0 < \theta < 1$.

(a) Let $T_k$ denote the UMVUE of $\theta^k$. Give an explicit formula for $T_k$, $k = 1, \ldots, n$.

(b) Suppose we are interested in estimating the odds ratio

$$r = \frac{\theta}{1 - \theta}.$$

Note that $r$ does not have an unbiased estimator (you don’t need to show this). What is the MLE $\hat{r}$ of $r$? What is the bias of $\hat{r}$?
5. Let $X_1, \ldots, X_n$ be iid Bernoulli($p$), and $Y = \sum_{i=1}^{n} X_i$. Assume the prior distribution on $p$ is distributed as beta with density

$$h(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}, \quad \alpha > 0, \beta > 0.$$ 

(a) Find the Bayes estimator of $p$ under the square error loss.

(b) Find the MSE of the Bayes estimator obtained in (a).
6. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density $f_\theta(x)$. Determine the UMP test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ when the density is

$$f_\theta(x) = \frac{1}{\theta} \exp(-x/\theta), \quad x > 0.$$
7. Let $X_1, \ldots, X_{n_1}$ be a random sample from the $N(\mu_1, \sigma^2)$ and let $Y_1, \ldots, Y_{n_2}$ be a random sample from the $N(\mu_2, \sigma^2)$ distribution which is independent of the first random sample. Consider the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

(a) Find the unrestricted MLE’s of $\mu_1, \mu_2$ and $\sigma^2$.

(b) Find the MLE’s of $\mu = \mu_1 = \mu_2$ and $\sigma^2$ under $H_0$.

(c) Show that
$$
\sum_{i=1}^{n_1}(x_i - \hat{\mu})^2 + \sum_{i=1}^{n_2}(y_i - \hat{\mu})^2
= \sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2 + n_1(\bar{x} - \hat{\mu})^2 + n_2(\bar{y} - \hat{\mu})^2.
$$

(d) Show that the likelihood ratio test is
$$
\phi(x) = \begin{cases} 
1 & \text{if } F > c, \\
0 & \text{if } F < c, 
\end{cases}
$$

$$
P_{H_0}(F > c) = \alpha, \text{ where }
F = \frac{n_1(\bar{x} - \hat{\mu})^2 + n_2(\bar{y} - \hat{\mu})^2}{[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2]/(n_1 + n_2 - 2)}.
$$