MATHEMATICAL STATISTICS

QUALIFYING EXAM

Fall 1999

NAME: ________________________________

The four problems you have attempted: ________________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

1. Let $X_n$ converge to $c$ in probability. If $h(x)$ is a continuous function at $x = c$, prove that $h(X_n)$ converges to $h(c)$ in probability.
2. Let $\overline{X}_n$ denote the mean of a random sample of size $n$ from a gamma distribution with pdf

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha - 1} e^{-x}, \quad x > 0, \quad \alpha > 0.$$ 

Find the limiting distribution of $\sqrt{n}(\overline{X}_n - \alpha)/\sqrt{\overline{X}_n}$. 
3. Suppose that $X_1, \ldots, X_n$ is a random sample from the exponential distribution with pdf

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0, \theta > 0.$$ 

Find the UMVUE of $e^{-k\theta}$ for fixed $k > 0$.

Hint: Define

$$U = \begin{cases} 1, & X_1 > k \\ 0, & X_1 \leq k. \end{cases}$$
4. Let $Y_n$ be the $n$th order statistic of a random sample of size $n$ from a distribution with pdf $f(x|\theta) = 1/\theta, \ 0 < x < \theta,$ zero elsewhere. Suppose the prior distribution on $\theta$ has pdf $\lambda(\theta) = \beta \alpha^\beta / \theta^{\beta+1}, \ \theta > \alpha,$ zero elsewhere, with $\alpha > 0$ and $\beta > 0.$ Find the Bayes estimator of $\theta$ under the squared error loss.
5. Let the distribution of $X$ be given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P_\theta(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\theta$</td>
</tr>
<tr>
<td>1</td>
<td>$2\theta$</td>
</tr>
<tr>
<td>2</td>
<td>$0.9 - 2\theta$</td>
</tr>
<tr>
<td>3</td>
<td>$0.1 - \theta$</td>
</tr>
</tbody>
</table>

where $0 < \theta < 0.1$. For testing $H_0 : \theta = 0.05$ against $H_1 : \theta > 0.05$ at level $\alpha = 0.05$, determine if the following test is UMP:

$\phi(1) = 0.5$, $\phi(0) = \phi(2) = \phi(3) = 0$. 
6. Find the likelihood ratio test of $H_0 : \sigma = 1$ versus $H_1 : \sigma \neq 1$ based on a sample $X_1, \ldots, X_n$ from a $N(\theta, \sigma \theta)$ family, where $\theta$ is unknown.