

Qualifying Exam (Spring 2001): Operations Research

You have 4 hours to do this exam. Do 2 out of problems 1,2,3. Do 2 out of problems 4,5,6. Do 3 out of problems 7,8,9,10,11,12,13,14. All problems are weighted equally. On the front page write clearly which seven problems you want graded.

Reminder: This exam is closed notes and closed books.

1). Consider the following LP:

$$\begin{aligned}
 \max \quad & z = 5x_1 + x_2 + 2x_3 \\
 & x_1 + x_2 + x_3 \leq 6 \\
 & 6x_1 + x_3 \leq 8 \\
 & x_2 + x_3 \leq 2 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Let the slack variables be s_i for the constraints. The optimal tableau is given below:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	1/6	0	0	5/6	7/6	9
0	0	1/6	0	1	-1/6	-5/6	3
0	1	-1/6	0	0	1/6	-1/6	1
0	0	1	1	0	0	1	2

- (a) Which variables are basic in the current BFS? What is the current B^{-1} ?
- (b) Find the dual to the original LP and its optimal solution. Make sure to give the optimal value of the dual variables and the objective function.
- (c) Find the range of values of the objective function coefficient for x_2 for which the current basis remains optimal.
- (d) Find the range of values of the RHS coefficient b_1 for which the current basis remains optimal.

2. Consider the LP $\{\min cx \mid Ax = b, x \geq 0\}$ A key result used in the Simplex method is that if $z_j - c_j \leq 0$ for all j , then the current basic feasible solution is optimal.

- (a) State a sufficient condition for the current basic feasible solution to be the *unique* optimal solution. Make sure to prove why this condition is sufficient.
- (b) Is the condition you stated in part (a) also necessary? (i.e., does every unique optimal basic feasible solution satisfy this condition?) Explain.

3. Suppose that the problem $\{\max cx \mid Ax = b, x \geq 0\}$ has a finite optimal solution. Let d be an arbitrary vector (of the same dimension as b). Show that if the problem $\{\max cx \mid Ax = d, x \geq 0\}$ has a feasible solution, then it has a finite optimal solution.

4. Suppose you have an access to the IRS computer and you started to look at all Social Security numbers stored in this computer. Let X_n denote the last digit of the n -th Social Security number and $Y_n = \sum_{k=1}^n X_k$. Find

$$\lim_{n \rightarrow \infty} P(Y_n \text{ is a multiple of } 11)$$

5. A machine is replaced when it fails. The distribution of the time until the failure of the machine is F . However, a newly arrived machine is allowed to work until time T (unless it has failed before and had been replaced) and then inspected. If it passes the inspection that it is allowed to work until the time of its failure, otherwise it is sent to the factory for replacement. The time for the replacement to arrive is S . The

probability of passing the inspection is p . Find the proportion of time when there is no working equipment at the site.

6. A group has four members. Each pair of the group at each given moment of time may or may not have a certain relationship with each other. If they have a relationship, we say that they are *linked*. E.g., being linked could mean that 2 members are communicating with each other. Suppose that any pair of unlinked individuals can become linked in a small time interval h with probability $\alpha h + o(h)$. Any pair of linked individuals will lose their link in a small time interval h with probability $\beta h + o(h)$. Any member of the group can have any number of links from 0 to 3. Creation or losing of a link happen independently of the same process for any other link. Let $X(t)$ denote the number of linked pairs in the group. Then $X(t)$ is a continuous time Markov chain.

- (a) Specify its Q -matrix (the infinitesimal generator).
- (b) What proportion of time all members of the group are linked to each other?

7. Let X_1, X_2, \dots be independent with distribution $U[0, 1]$ (i.e. uniformly distributed on the interval $[0, 1]$).

- (a) Show that as $n \rightarrow \infty$, $\max\{X_1, \dots, X_n\} \rightarrow 1$ almost surely.
- (b) Show that as $n \rightarrow \infty$, $n(1 - \max\{X_1, \dots, X_n\})$ tends in distribution to an exponential distribution.

8. Let (Ω, \mathcal{F}, P) be a probability space, X , and Y random variables in $L^2(\Omega, \mathcal{F}, P)$. Suppose also that $E[X|Y] = Y$ and $E[Y|X] = X$. Show then that $X = Y$ (almost surely).

9. We want to find a minimum spanning tree on a graph G with weights on the edges $w(e)$. Consider the following *reverse greedy* algorithm:

Sort the edges in decreasing order of their weights $w_{e_1} \geq w_{e_2} \geq \dots \geq w_{e_m}$.
 Let $G' := G$
 For $k = 1$ to m do:
 If $G' \setminus e_k$ is connected then set $G' := G' \setminus e_k$
 end

Prove that at the termination of this algorithm G' is a minimum spanning tree.

10. Given the minimum cost flow problem in a directed network $G = (N, A)$

$$\begin{aligned} \min \quad & z(x) = \sum_{(i,j) \in A} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_j x_{ij} - \sum_j x_{ji} = b(i) \quad \text{for all } i \in N \\ & 0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i,j) \in A \end{aligned}$$

- (a) State the dual maximization problem.
- (b) Show that the minimum value of $z(x)$ is greater than or equal to the maximum value of the dual objective function. Hint: You may solve this problem with a network flow specific analysis or a general mathematical programming approach.
- (c) Define the residual network $G^* = (N, A^*)$ for a particular feasible solution x^* in the min cost flow problem with each arc $(i, j) \in A$ replaced by arcs (i, j) and (j, i) in A^* with residual capacities $r_{ij} = u_{ij} - x_{ij}^*$ and $r_{ji} = x_{ij}^*$, and with c_{ij} unchanged and $c_{ji} = -c_{ij}$. Show that x^* is optimal if G^* has no negative cost cycle.

11. (a) Briefly explain what the advantage might be in having a modulus that is a power of 2 in a linear congruential random number generator.

(b) Briefly explain what the disadvantage might be.

(c) Use the linear congruential method with: multiplier $A = 8$, additive constant $C = 47$, modulus $m = 100$ and initial seed value $X_0 = 27$ to generate the next three uniformly distributed random numbers in the range $[0,1)$.

(d) Name a method by which the uniformly distributed variates of part (c) above can be transformed into exponentially distributed variates. Briefly explain the logic behind the method.

(e) Apply the method to generate 3 exponentially distributed variates from a distribution with mean 2, using the 3 values you obtained in part (c).

12. Four independent replications are made of a single server queuing system starting with an empty system. Each replication simulates the first 2 hours of the system's operation. Two performance metrics are collected from each replication: the server utilization, and the mean time spent by a customer in the system:

Replication	Utilization	Mean time in system (minutes)
1	0.808	3.74
2	0.875	4.53
3	0.708	3.84
4	0.842	3.98

(a) Calculate an estimate and a 95% level confidence interval for the mean time in system based on the four replications. Select the most appropriate value from the following for your calculations:

$z_{0.975} = 1.96$ (standard normal distribution)

$t_{3;0.975} = 3.182$ (student's t-distribution with 3 degrees of freedom)

$t_{4;0.975} = 2.776$ (student's t-distribution with 4 degrees of freedom)

(b) Suppose we want the 95% confidence interval to have a half width of no more than ± 0.4 minutes. Explain how you would use the results of part (a) to estimate the number of replications we would need, and calculate the estimate.

(c) Returning to part (a), suppose we were also to calculate a 95% confidence interval for the server utilization. What would the overall confidence level of the two confidence intervals calculated be? Why? (in other words, justify your answer by explaining the logic underlying it). What would we need to do to obtain an overall confidence level of 95%?

13. Let $P = (p_1, p_2, \dots, p_n)$ be a given polygonal chain (possibly self-intersecting – we do NOT assume that it is simple; it may or may not be closed (i.e., p_1 may equal p_n)). (The vertices of the chain are the points p_i , the edges are the line segments p_1p_2, p_2p_3, \dots)

(a). Short answer questions (no proofs needed):

(i). How efficiently can the convex hull, $CH(P)$, be computed?

(ii). How efficiently can one test if P intersects itself? (assume that it is nondegenerate – no coincident vertices and no three collinear vertices)

(b). How efficiently can one determine if P is (exactly) the boundary of a convex polygon?

(c). Give an efficient method to determine if there exists a line, ℓ , of any orientation, that stabs (intersects) every segment $(p_i p_{i+1})$ of P . (Hint: It can be done in $O(n)$ time. Try to think of a way to achieve $O(n)$; if you cannot, then do it as efficiently as you can.)

14. Let $\mathcal{T} = \{T_1, \dots, T_n\}$ be an arbitrary set of (possibly overlapping) triangles in the plane.

(a). What is the worst-case complexity of their arrangement? (Give upper and lower bounds; justify briefly.)

(b). Let the *thickness* of \mathcal{T} be the maximum number of triangles overlapping at a single point. Describe an efficient algorithm to compute the thickness, and state its time complexity.

(c). Explain how to preprocess \mathcal{T} so that if I give you an axis-parallel rectangle query, you can quickly report the set of triangles that are *fully* contained in the query rectangle. Give the space, preprocessing and query bounds. You may treat standard methods of range search, point location, etc, as black boxes.