

Qualifying Exam (Spring 2002): Operations Research

You have 4 hours to do this exam. Do 2 out of problems 1,2,3. Do 2 out of problems 4,5,6. Do 3 out of problems 7,8,9,10,11,12. All problems are weighted equally. On the front page write clearly which seven problems you want graded.

Reminder: This exam is closed notes and closed books.

- 1). Let x be a BFS for a maximization LP. Prove the following:
 - (a) If the reduced cost \bar{c}_i of every non basic variable is positive, then x is the unique optimal solution.
 - (b) If x is the unique optimal solution and is non degenerate, then the reduced cost of every non basic variable is positive.

- 2). Let A be a given matrix. Show that exactly one of the following alternatives must hold:
 - (a) There exists some $x \neq 0$ such that $Ax = 0, x \geq 0$
 - (b) There exists some y such that $A^t y > 0$.

- 3). Consider the following LP:

$$\begin{aligned}
 \max \quad & z = 4x_1 + x_2 \\
 & 3x_1 + x_2 \geq 6 \\
 & 2x_1 + x_2 \geq 4 \\
 & x_1 + x_2 = 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Let the excess variables be ϵ_i for the constraints, and the artificial variables be a_i . The optimal tableau is given below (solved using the big M method):

z	x_1	x_2	ϵ_1	ϵ_2	a_1	a_2	a_3	RHS
1	0	3	0	0	M	M	$M + 4$	12
0	1	1	0	0	0	0	1	3
0	0	2	1	0	-1	0	3	3
0	0	1	0	1	0	-1	2	2

- (a) Which variables are basic in the current BFS? What is the current B^{-1} ?
 - (b) Find the dual to the original LP and its optimal solution. Make sure to give the optimal value of the dual variables and the objective function.
 - (c) Find the range of values of the objective function coefficient for x_2 for which the current basis remains optimal.
 - (d) Find the range of values of the RHS coefficient b_1 (currently 6) for which the current basis remains optimal.
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- 4). Bears come to the village at Poisson rate λ . They are caught (immediately) by the rangers and kept at a cost of $\$c$ per day for each bear. When Q bears are caught they are immediately transported back to the woods at a cost of $\$d$ for the whole batch.
- Find the policy (that is find Q) which minimizes the average per unit time cost of "bear management".
- Hint: Solve the problem as if Q can be any real number and then look for the integer close to the solution you obtain.
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- 5). Consider a connected graph with N vertices $\{1, 2, \dots, N\}$. Let $d(i)$ denote the degree of vertex i , that is $d(i)$ is the number of edges incidental to vertex i . A particle moves from vertex to vertex of the graph each time according to the following rule.

If at the n -th step the particle is at the vertex i , then each edge leading from this vertex is chosen with equal probability ($=\frac{1}{d(i)}$) and the particle moves to the vertex where the chosen edge leads. Let X_n denote the position (vertex) of the particle at the n -th step.

- (a) Show that X_n is a time reversible Markov chain.
 (b) Find the steady state probabilities for this Markov chain.

Hint: Try to do (a) and (b) simultaneously.

6). There are N machines and N repairmen. Each machine is working for an exponential time with parameter λ . When it is broken, it goes to the first available repairman. The repair time is exponential with parameter μ . Let $X(t)$ be the number of machines functioning at time t .

- (a) Write the Q matrix for the continuous time Markov chain $X(t)$.
 (b) Find the steady state distribution for this Markov chain.
 (c) Write a normal approximation for the distribution in (b) when N is large.

7). Let Ω be the set of infinite sequences of 0's and 1's:

$$\Omega = \{\omega = (\omega_1, \omega_2, \omega_3, \dots) \mid \omega_n = 0 \text{ or } 1, \text{ for all } n\}$$

with its usual σ algebra. For $n = 1, 2, \dots$ let X_n be the random variable on Ω defined as:

$$X_n(\omega_1, \omega_2, \omega_3, \dots) = \omega_n$$

P_1 is the probability measure on Ω defined by saying that the sequence of random variables $\{X_n\}_1^\infty$ is i.i.d and $P_1(X_n = 0) = 1/2$.

(a) Let X be the random variable $\sum_{n=1}^{\infty} \frac{X_n}{2^n}$. Determine the characteristic function of X and show that it equals the characteristic function of the uniform distribution on the interval $[0, 1]$.

(b) Let P_2 be the probability measure defined on Ω defined by saying that the sequence of random variables $\{X_n\}_1^\infty$ is i.i.d and $P_2(X_n = 0) = 1/3$. Construct an event A with the property: $P_1(A) = 1$ and $P_2(A) = 0$.

8). The (weak and strong) Laws of Large Numbers and the Central Limit theorem have as standard hypothesis that random variables are independent and identically distributed. These theorems can be generalized in various ways. The goal of this problem is to investigate what happens when the random variables are 'almost' independent (in the sense defined below).

Let $\{X_n\}_{n=1}^\infty$ be a sequence of identically distributed random variables with the uniform distribution on the interval $[-1, 1]$. (in particular the mean is zero, and all moments are finite). Assume that there exists a positive number α , less than one, so that for all n and m :

$$|E(X_n X_m)| \leq \alpha^{n-m}$$

This hypothesis means that the outcomes X_n and X_m are nearly uncorrelated when n and m are very different.

(a) Show that the sequence of random variables $\frac{\sum_{i=1}^n X_i}{n}$ satisfies the weak law of large numbers.

(b) Show that the sequence of random variables $\frac{\sum_{i=1}^n X_i}{n}$ satisfies the strong law of large numbers.

9). Recall the Floyd Warshall algorithm for finding all pairs shortest paths in a directed graph with general costs: Define $d^k[i, j]$ to be the length of a shortest path from i to j using only nodes $1, 2, \dots, k-1$ as

intermediate nodes. Compute $d^{k-1} = \min\{d^k[i, j], d^k[i, k] + d^k[k, j]\}$. Prove: A negative cost cycle exists iff there exists a node i , such that $d^{n-1}[i, i] < 0$ (where n is the number of nodes in the graph).

10). (a) A maximum flow problem with several source nodes S' and several sink nodes T' ($S' \cap T' = \emptyset$) is to maximize $\sum_{i \in S', j \in T'}$ flow from i to j , where for each node not in $S' \cup T'$ we have conservation of flow, and for each arc, the total flow on the arc does not exceed its capacity. Show how to transform the maximum flow problem with several source nodes and several sink nodes into a problem with only one source node and one sink node.

(b) Let S' and T' be two non empty node disjoint sets in a directed graph G . Describe a method for determining the maximum number of arc disjoint paths from S' to T' (i.e., each path can start at any node of S' and end at any node of T'). State and prove a theorem for this version of Menger's Theorem: The maximum number of arc disjoint paths from S' to T' is equal to the minimum ...

11). Let S denote a set of n points in the plane. Assume that S is contained in the unit-radius disk centered on the origin. Let R denote the intersection of all unit-radius disks (centered anywhere in the plane) that fully contain S . Thus, it is easy to see that the boundary of R , ∂R , consists of a finite union of circular arcs (from circles of radius 1); the number of such arcs is the *complexity* of ∂R . Let $S' = S \cap \partial R$ denote the subset of points of S that appear on the boundary of R . Assume nondegeneracy: No two points of S have the same x - or y -coordinate and no three points of S lie on a common unit-radius circle.

(a). Give an example to show that the complexity of ∂R may be only $O(1)$, while the convex hull complexity (number of sides of $CH(S)$) is $h = \Omega(n)$.

(b). Prove or disprove: The point $p \in S$ having smallest y -coordinate must be in S' .

(c). Explain how one can find one point $p \in S'$ (any point) in time $O(n)$.

(d). Sketch how one can devise an efficient algorithm to compute the set S' of points on ∂R , in sorted order around the boundary ∂R . How efficient is your algorithm (in terms of big-Oh notation)? Try to obtain the best efficiency you can.

12). Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a set of n axis-aligned (possibly overlapping) rectangles in the plane. Assume nondegeneracy: No two x - or y -coordinates of sides of the rectangles are identical.

(a). Consider the arrangement induced by \mathcal{R} in the plane. Let n_v be the number of vertices (0-faces) in the arrangement. What is maximum/minimum possible value of n_v ? (Give exact bounds, not just big-Oh bounds.)

(b). Let the *thickness* of \mathcal{R} be the maximum number of rectangles overlapping at a single point.

(i). Describe an efficient sweepline algorithm to compute the thickness, and state its time complexity. Be explicit about what data structures you use and how you update them. What is the worst-case running time? (you may use big-Oh notation here)

(ii). Argue a lower bound of $\Omega(n \log n)$ on computing the thickness.